

# TRANSVERSALITY AND SUPER-RIGIDITY FOR MULTIPLY COVERED HOLOMORPHIC CURVES

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**ABSTRACT.** We develop new techniques to study regularity questions for moduli spaces of pseudoholomorphic curves that are multiply covered. Among the main results, we show that unbranched multiple covers of closed holomorphic curves are generically regular, and simple index 0 curves in dimensions greater than four are generically super-rigid, meaning their branched covers form connected components of the moduli space. We also establish partial results on super-rigidity in dimension four and regularity of branched covers, and briefly discuss the outlook for bifurcation analysis. The proofs are based on a representation-theoretic splitting of Cauchy-Riemann operators with symmetries.

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## 1. INTRODUCTION

The issue of transversality in Gromov's theory of pseudoholomorphic curves [Gro85] has always been problematic, and has attracted renewed interest in recent years. While many powerful symplectic invariants such as Gromov-Witten theory, Hamiltonian Floer homology and symplectic field theory are based on holomorphic curves, most of them run into severe technical complications unless multiply covered curves can be excluded, thus necessitating rather sophisticated techniques that typically replace the standard nonlinear Cauchy-Riemann equation by an abstract perturbation, see e.g. [FO99, LT98, Rua99, Sie, CM07, IPa, HWZ, Par16]. Aside from the technical challenges that these methods pose, they are non-ideal for many applications: for instance abstract perturbations destroy intersection theory in symplectic 4-manifolds, and in Calabi-Yau 3-folds they obscure information that one might hope to find in the geometric relationship between simple curves and their multiple covers, as exemplified by the Gopakumar-Vafa conjecture [GV, Pan99, BP01, BP08, IPb].

The motivating principle of this paper is in some sense orthogonal to that of abstract perturbations: our aim will be to extend the transversality theory for the *standard* pseudoholomorphic curve equation as far as it can reasonably be pushed, i.e. to prove transversality when it is possible, and in other cases to isolate the precise phenomena which make it impossible and explain what is true instead. Let us start by singling out two situations in which this program is not obviously hopeless.

**Example 1.1.** If  $u : (\Sigma, j) \rightarrow (M, J)$  is a closed  $J$ -holomorphic curve and  $\varphi : (\tilde{\Sigma}, \tilde{j}) \rightarrow (\Sigma, j)$  is an *unbranched* cover of closed connected Riemann surfaces with degree  $d \in \mathbb{N}$ , then the virtual dimensions of the moduli spaces containing  $u$  and  $u \circ \varphi : (\tilde{\Sigma}, \tilde{j}) \rightarrow (M, J)$ , also known as the *indices* of these two curves, are related by

$$\text{ind}(u \circ \varphi) = d \cdot \text{ind}(u).$$

Since  $\text{ind}(u \circ \varphi)$  is then nonnegative whenever  $\text{ind}(u) \geq 0$ , there is no obvious reason why  $u \circ \varphi$  could not achieve transversality generically, but traditional methods in the theory of  $J$ -holomorphic curves do not prove this except when  $u \circ \varphi$  is simply covered, or in certain 4-dimensional cases [HLS97], or more recently, when  $\text{ind}(u) = 0$  if a sufficiently large space of perturbed almost complex structures is allowed [GW].

**Example 1.2.** Suppose  $u : (\Sigma, j) \rightarrow (M, J)$  is a closed simply covered curve with index 0 and  $\varphi : (\tilde{\Sigma}, \tilde{j}) \rightarrow (\Sigma, j)$  is a branched cover of closed connected Riemann surfaces with degree  $d \in \mathbb{N}$  and  $Z(d\varphi) \geq 0$  as the algebraic count of branch points. Then combining the Riemann-Hurwitz formula

$$(1.1) \quad -\chi(\tilde{\Sigma}) + d \cdot \chi(\Sigma) = Z(d\varphi)$$

with the standard index formula for closed holomorphic curves gives the relation

$$(1.2) \quad \text{ind}(u \circ \varphi) = d \cdot \text{ind}(u) - (n - 3)Z(d\varphi) = -(n - 3)Z(d\varphi),$$

where  $\dim_{\mathbb{R}} M = 2n$ . This shows that  $u \circ \varphi$  can live in a space of negative virtual dimension when  $\dim M > 6$  and thus cannot generally achieve transversality, even though it will survive perturbations of  $J$  as long as  $u$  also does. It is interesting however to observe that  $u$  must be immersed if  $J$  is generic, so it has a well-defined normal bundle  $N_u \rightarrow \Sigma$ ,

and restricting the linearized Cauchy-Riemann operators for  $u$  and  $u \circ \varphi$  to the normal bundle and its pullback gives operators  $\mathbf{D}_u^N$  and  $\mathbf{D}_{u \circ \varphi}^N$  with indices related by

$$\mathrm{ind}(\mathbf{D}_{u \circ \varphi}^N) = d \cdot \mathrm{ind}(\mathbf{D}_u^N) - (n-1)Z(d\varphi) = -(n-1)Z(d\varphi).$$

The latter is always nonpositive, so  $\mathbf{D}_{u \circ \varphi}^N$  can be injective, and this condition has a geometric meaning: as shown in Proposition 1.3 below, it implies that  $u \circ \varphi$  can never be the limit of a sequence of somewhere injective curves. In fact, the only other curves near  $u \circ \varphi$  are other branched covers of the form  $u \circ \varphi'$  for  $\varphi'$  near  $\varphi$ , and the cokernels of the operators  $\mathbf{D}_{u \circ \varphi}^N$  define an obstruction bundle over the space of branched covers which can be used to compute Gromov-Witten invariants. This phenomenon is known as *super-rigidity*, see Definition 2.4.

Considerable interest in super-rigidity has been motivated by the study of Gromov-Witten invariants in Calabi-Yau 3-folds, notably the Gopakumar-Vafa conjecture. Bryan and Pandharipande observed in [BP01] that while super-rigidity is a quite delicate question in the *algebraic* geometry of Calabi-Yau 3-folds, it might reasonably be expected to hold generically in the symplectic setting. Theorem A below states that this is true, in fact for generic  $J$ , *all* somewhere injective closed  $J$ -holomorphic curves of index 0 in symplectic manifolds of dimension greater than four (and also curves of low genus in dimension four) are super-rigid. Complementary to this, we will see in Theorem B that everything one could hope for in the setting of Example 1.1 is also true, and we will also be able to prove some transversality results for branched covers (Theorem C). All of these results follow from a general picture of Cauchy-Riemann type operators with symmetries described in §2.2, which has its origins in Taubes's work on the Gromov invariant of symplectic 4-manifolds [Tau96a]. While the results in this paper focus specifically on closed holomorphic curves, there is no obvious obstruction to applying the same techniques to study punctured curves in symplectic cobordisms, which we expect to have interesting applications in symplectic field theory [EGH00] and Embedded Contact Homology [Hut14]. A few special cases of super-rigidity in the punctured case have previously been observed in [Wen10, Fab13]; those examples were restricted to dimension four, but the results of the present article suggest that super-rigidity is likely to hold in considerably greater generality.

**1.1. Super-rigidity and transversality theorems.** To state the main results, assume  $(M, \omega)$  is a symplectic manifold with

$$\dim M = 2n \geq 4,$$

and  $J_{\mathrm{fix}}$  is a smooth almost complex structure that is **compatible** with  $\omega$ , meaning that  $\omega(\cdot, J_{\mathrm{fix}}\cdot)$  defines a Riemannian metric on  $M$ . We fix also an open subset  $\mathcal{U} \subset M$  with compact closure, and consider the space

$$\mathcal{J}(M, \omega; \mathcal{U}, J_{\mathrm{fix}})$$

of smooth  $\omega$ -compatible almost complex structures on  $M$  that match  $J_{\mathrm{fix}}$  outside of  $\mathcal{U}$ , with its natural  $C^\infty$ -topology. Note that the existence of a symplectic form on  $M$  is not required for any of the arguments in this paper, but we are including it in the setup since it is important in applications—all results could alternatively be stated and proved for

the larger space of  $\omega$ -tame almost complex structures, or for arbitrary almost complex structures on a smooth (not necessarily symplectic) manifold.

Following the usual convention among symplectic topologists, we will say that a subset of a topological space is a **Baire subset** if it is comeager, i.e. it is a countable intersection of open and dense subsets. The intersection of a countable sequence of Baire subsets is again a Baire subset, and by the Baire category theorem, any Baire subset of a complete metric space is dense. We will say that a given property is true **generically** (e.g. for generic  $J$ ) whenever there exists a Baire subset of the space of all admissible data (e.g. in  $\mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$ ) such that the property holds for all choices of data in that subset.

Given  $J \in \mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$ , a closed connected Riemann surface  $(\Sigma, j)$  and a  $J$ -holomorphic curve  $u : (\Sigma, j) \rightarrow (M, J)$ , the **index** of  $u$  is the integer

$$(1.3) \quad \text{ind}(u) = (n - 3)\chi(\Sigma) + 2c_1(u),$$

where we abbreviate  $c_1(u) := \langle c_1(TM, J), [u] \rangle$ ,  $[u] := u_*[\Sigma] \in H_2(M)$ . A closed and connected  $J$ -holomorphic curve  $\tilde{u} : (\tilde{\Sigma}, \tilde{j}) \rightarrow (M, J)$  is said to be a  $(d\text{-fold})$  **multiple cover** of  $u$  if  $\tilde{u} = u \circ \varphi$  for some holomorphic map  $\varphi : (\tilde{\Sigma}, \tilde{j}) \rightarrow (\Sigma, j)$  of degree  $d \geq 2$ , and  $u$  is called **simple** if it is nonconstant and is not a multiple cover of any other curve.

The notion of super-rigidity was outlined already in Example 1.2; see Definition 2.4 for a more precise formulation. We will also use the term *Fredholm regular* to refer to the standard notion of transversality for moduli spaces of unparametrized  $J$ -holomorphic curves, cf. Proposition 2.3 below. In each of the following theorems,  $(M, \omega)$  is a symplectic manifold of dimension  $2n$  with a compatible almost complex structure  $J_{\text{fix}}$ , and  $\mathcal{U} \subset M$  is an open subset with compact closure.

**Theorem A** (super-rigidity). *If  $\dim M \geq 6$ , then there exists a Baire subset  $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$  such that for all  $J \in \mathcal{J}_{\text{reg}}$ , every simple  $J$ -holomorphic curve of index 0 that intersects  $\mathcal{U}$  is super-rigid. Moreover, this result also holds when  $\dim M = 4$  for all simple index 0 curves of genus 0 or 1.*

Super-rigidity has a number of well-known consequences, which are especially important in the case  $\dim M = 6$ . These are based partly on the observation that the space of all covers of super-rigid curves is an open and closed subset of the ambient moduli space of  $J$ -holomorphic curves—we will prove the following statement in Appendix B, see also Corollary 2.2.

**Proposition 1.3.** *Suppose  $(M, J_k)$  is a sequence of almost complex manifolds with  $J_k \rightarrow J_\infty$  in  $C^\infty$  on some compact subset containing a super-rigid  $J_\infty$ -holomorphic curve  $u_\infty : (\Sigma, j_\infty) \rightarrow (M, J_\infty)$ . Then for sufficiently large  $k$ , there exists a sequence of  $J_k$ -holomorphic curves  $u_k : (\Sigma, j_k) \rightarrow (M, J_k)$  with  $j_k \rightarrow j_\infty$  and  $u_k \rightarrow u_\infty$  in  $C^\infty$ , and if  $v_k$  is any sequence of smooth closed  $J_k$ -holomorphic curves Gromov-convergent to a stable nodal  $J_\infty$ -holomorphic curve with image contained in  $u_\infty(\Sigma)$ , then for all  $k$  sufficiently large, every  $v_k$  is either a biholomorphic reparametrization or a multiple cover of  $u_k$ .*

Applying Gromov compactness and the standard implicit function theorem for simple curves, plus the fact that simple  $J$ -holomorphic curves are generically embedded and disjoint from each other in dimensions greater than four, this implies:

**Corollary 1.4.** *For generic compatible  $J$  in a closed symplectic 6-manifold  $(M, \omega)$ , there exist for each  $g \geq 0$  at most finitely many distinct simple  $J$ -holomorphic curves  $u$  of genus  $g$  with  $c_1(u) = 0$ .  $\square$*

Using results of Zinger [Zin11] and Lee-Parker [LP12], Theorem A also implies that the space of branched covers of an embedded index 0 curve generically admits a well-defined obstruction bundle which can be used to compute Gromov-Witten invariants. In particular, if  $\dim M \geq 6$  and  $u : (\Sigma, j) \rightarrow (M, J)$  is an embedded  $J$ -holomorphic curve with index 0, one can apply [Zin11, Theorem 1.2] with no marked point constraints to study the space of  $J$ -holomorphic curves with image in  $u(\Sigma)$ , so that Theorem A establishes hypothesis (b) in Zinger's result, implying that the cokernels of the normal operators  $\mathbf{D}_{\tilde{u}}^N$  for  $\tilde{u} = u \circ \varphi$  varying in the (compactified) space  $\overline{\mathcal{M}}_h(d; u)$  of  $d$ -fold branched covers of  $u$  with arithmetic genus  $h$  form a well-defined and oriented orbundle

$$\mathcal{O}b \rightarrow \overline{\mathcal{M}}_h(d; u)$$

with  $\text{rank}_{\mathbb{R}} \mathcal{O}b = (n-1)Z(d\varphi)$ . Here  $Z(d\varphi)$  again denotes the count of critical points of  $\varphi$ , which can be expressed in terms of  $d, g$  and  $h$  via the Riemann-Hurwitz formula. This is most interesting in the 6-dimensional case, since  $n = 3$  then means that  $\text{rank}_{\mathbb{R}} \mathcal{O}b = 2Z(d\varphi) = \dim \overline{\mathcal{M}}_h(d; u)$ , and the count of solutions to an abstract perturbation of the holomorphic curve equation can then be computed as the Euler number of this bundle.

Recall that  $(M, \omega)$  is called a **symplectic Calabi-Yau 3-fold** if  $\dim M = 6$  and  $c_1(TM, \omega) = 0$ . In this case Corollary 1.4 applies to *all* simple curves in  $M$ , with the consequence that the Gromov-Witten invariants can be reduced to finite sums of *local Gromov-Witten invariants* for embedded curves, computed in terms of the Euler classes of the obstruction bundles mentioned above. This has well-known applications to the Gopakumar-Vafa conjecture, see [Pan99, BP01, BP08, IPb].

We next state two results on transversality for multiple covers.

**Theorem B** (transversality). *There exists a Baire subset  $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$  such that for all  $J \in \mathcal{J}_{\text{reg}}$ , for every simple  $J$ -holomorphic curve  $u : (\Sigma, j) \rightarrow (M, J)$  intersecting  $\mathcal{U}$  and every unbranched holomorphic cover  $\varphi : (\tilde{\Sigma}, \tilde{j}) \rightarrow (\Sigma, j)$  of closed Riemann surfaces, the  $J$ -holomorphic curve  $u \circ \varphi : (\tilde{\Sigma}, \tilde{j}) \rightarrow (M, J)$  is Fredholm regular.*

*Remark 1.5.* The case  $\text{ind}(u) = 0$  of Theorem B has been proved previously in [GW], though with stronger assumptions: for technical reasons, it was necessary in that paper to assume that  $u(\Sigma)$  is *contained entirely* in  $\mathcal{U}$ , and in dimension four also to allow perturbations of  $J$  that are  $\omega$ -tame but not necessarily  $\omega$ -compatible. The present paper uses a completely different approach to the transversality problem and is thus able to remove these restrictions.

It is generally harder to achieve transversality for covers  $u \circ \varphi$  with branch points, e.g. the index relation (1.2) shows that  $\text{ind}(u \circ \varphi)$  can easily become negative in dimensions greater than six. More seriously, if  $u$  is regular, then one can always find a smooth family of other multiple covers near  $u \circ \varphi$  obtained by varying both  $u$  and  $\varphi$  in their respective moduli spaces; since the latter lives in a space of real dimension  $2Z(d\varphi)$ , the condition

$$\text{ind}(u \circ \varphi) \geq \text{ind}(u) + 2Z(d\varphi)$$

is evidently necessary in order for  $u \circ \varphi$  to be Fredholm regular. Observe that if  $\varphi$  has  $r \geq 0$  critical values, then this condition is satisfied whenever  $\text{ind}(u) \geq (n-1)r$ : indeed, each critical value is the image of at most  $d-1$  branch points (counted algebraically), so we have  $Z(d\varphi) \leq (d-1)r$  and (1.2) implies

$$\begin{aligned} \text{ind}(u \circ \varphi) &= \text{ind}(u) + (d-1)\text{ind}(u) - (n-3)Z(d\varphi) \\ &\geq \text{ind}(u) + (n-1)Z(d\varphi) - (n-3)Z(d\varphi) = \text{ind}(u) + 2Z(d\varphi). \end{aligned}$$

The next result states that the condition  $\text{ind}(u) \geq (n-1)r$  is also, in some sense, sufficient.

**Theorem C** (transversality). *There exists a Baire subset  $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$  such that the following holds for all  $J \in \mathcal{J}_{\text{reg}}$ . Suppose  $u : (\Sigma, j) \rightarrow (M, J)$  is a simple  $J$ -holomorphic curve intersecting  $\mathcal{U}$  and satisfying*

$$\text{ind}(u) \geq (n-1)r$$

*for some integer  $r \geq 0$ , and  $\varphi : (\tilde{\Sigma}, \tilde{j}) \rightarrow (\Sigma, j)$  is a holomorphic branched cover of closed connected Riemann surfaces with  $r$  distinct critical values. Then there exists a  $J$ -holomorphic curve and a holomorphic branched cover*

$$u_\epsilon : (\Sigma, j_\epsilon) \rightarrow (M, J) \quad \text{and} \quad \varphi_\epsilon : (\tilde{\Sigma}, \tilde{j}_\epsilon) \rightarrow (\Sigma, j_\epsilon)$$

*such that  $u_\epsilon$ ,  $\varphi_\epsilon$ ,  $j_\epsilon$  and  $\tilde{j}_\epsilon$  are arbitrarily  $C^\infty$ -close to  $u$ ,  $\varphi$ ,  $j$  and  $\tilde{j}$  respectively, and  $u_\epsilon \circ \varphi_\epsilon : (\tilde{\Sigma}, \tilde{j}_\epsilon) \rightarrow (M, J)$  is Fredholm regular.*

Note that whenever  $\text{ind}(u \circ \varphi)$  is *strictly* greater than  $\text{ind}(u) + 2Z(d\varphi)$ , one can combine this result with the implicit function theorem to deduce the existence of simple  $J$ -holomorphic curves that are small perturbations of multiple covers of  $u$ .

*Remark 1.6.* As will be clear from the proof of Theorem C, one can also arrange for the perturbed branched cover  $\varphi_\epsilon$  to have the same number of critical points and critical values as  $\varphi$ , i.e. it is a nearby point in the same moduli space of holomorphic branched covers with fixed *branching data*, cf. §2.2.

The proofs of these theorems are inspired by the work of Taubes [Tau96a], whose definition of the Gromov invariant for symplectic 4-manifolds required a special case of Theorem B along with related bifurcation-theoretic results (cf. §2.4) for multiply covered holomorphic tori. Roughly speaking, the idea is to study the local structure of spaces of the form

$$(1.4) \quad \mathcal{M}(k, c) := \{ \tilde{u} = u \circ \varphi \mid \dim \ker \mathbf{D}_{\tilde{u}}^N = k \text{ and } \dim \text{coker } \mathbf{D}_{\tilde{u}}^N = c \},$$

where  $k, c \geq 0$  are fixed integers,  $u$  varies in the moduli space of simple  $J$ -holomorphic curves and  $\varphi$  varies in the moduli space of holomorphic branched covers. Ideally, one would like to show that these spaces are smooth manifolds for generic  $J$ , and to compute their codimensions in the space of pairs  $(u, \varphi)$ . This turns Theorems A and B into “dimension counting” problems, as whenever one can show that the codimension of  $\mathcal{M}(k, c)$  is larger than the dimension of the ambient space, one concludes that either  $\ker \mathbf{D}_{\tilde{u}}^N$  or  $\text{coker } \mathbf{D}_{\tilde{u}}^N$  cannot be nontrivial. This discussion is oversimplified in at least three respects: first, we will not be able to find any nice structure on  $\mathcal{M}(k, c)$  if  $\varphi$  varies in the space of *all* branched covers, but it will help to confine it to certain substrata of that space in which all branch points have prescribed branching orders. For similar reasons, it will also help to confine



$u$  to substrata in which its number of critical points and their orders are constrained, and this is easily done. More seriously, the space  $\mathcal{M}(k, c)$  as sketched above can have different codimensions on different components, as its codimension depends intricately on symmetry information which is ignored in (1.4). We will therefore need to define a more elaborate version of  $\mathcal{M}(k, c)$  which depends on a splitting of the operator  $\mathbf{D}_u^N$  into summands corresponding to irreducible representations of the (generalized) symmetry group of the cover. This idea is borrowed directly from [Tau96a], though the details are somewhat more involved since, in contrast to the case of unbranched covers of tori, we cannot assume that all covers are regular or that their symmetry groups are abelian. We will see that once the formalism is developed in sufficient generality, it “breaks the symmetry” of  $\mathbf{D}_u^N$  enough to make dimension counting arguments much more effective.

*Remark 1.7.* A slightly different variation on the ideas in [Tau96a] has been implemented by Eftekhary to prove a partial result toward super-rigidity in dimension six, see [Eft].

Here is an outline of the rest of the paper.

After establishing some standard definitions and notation, §2 will further elucidate the ideas sketched above and formulate a precise version of the statement that  $\mathcal{M}(k, c)$  from (1.4) is a smooth submanifold, Theorem D. This will then be used as a black box to prove Theorems A, B and C in §2.3, followed in §2.4 by a brief informal discussion of bifurcation theory. The remainder of the paper is then devoted to the proof of Theorem D. In §3, we explain the splitting construction for Cauchy-Riemann operators with symmetries and prove some lemmas based on a mixture of elliptic regularity for punctured Cauchy-Riemann operators, topology of covering spaces, and representation theory of finite groups. The summands in the splitting are also Cauchy-Riemann operators, whose indices are a somewhat delicate computation, carried out in §4. In §5 we prove the special unique continuation result that is required for the Sard-Smale argument in §6, which completes the proof of Theorem D. Finally, §7 deals with super-rigidity in the four-dimensional case, which is something of an anomaly and requires different techniques based on intersection theory. The two appendices prove a pair of results that are considered “standard” and yet, in this author’s experience, seem sufficiently badly understood among experts to warrant some discussion; the proofs require a few ideas that will in any case be useful elsewhere in the paper.

**1.2. Apologies and acknowledgements.** The super-rigidity problem has a slightly troubled history, and as the author of a new paper on the subject, it would behoove me at this point to apologize for having caused some of that trouble: I am aware of three previous attempts to prove some version of Theorem A which were later either withdrawn or revised to prove much weaker statements, and I was an author of one of them (the original version of [GW]). With that in mind, I would sympathize with any reader’s inclination to greet the present paper with a dose of skepticism, though it seems worth pointing out that rather than being an attempt to rescue the (probably unrescuable) proof originally attempted in [GW], the approach taken in this paper has almost nothing in common with the previous one, other than the considerable debt that both of them owe to the ideas of Taubes [Tau96b, Tau96a].

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## 2. THE MAIN IDEA

**2.1. Some definitions.** Let us now fix some notation and definitions that will be essential in the rest of the paper.

Given integers  $g, m \geq 0$  and a homology class  $A \in H_2(M)$ , the moduli space of **unparametrized  $J$ -holomorphic curves**  $\mathcal{M}_{g,m}(A, J)$  can be defined as the set of equivalence classes of tuples  $(\Sigma, j, \Theta, u)$  where  $(\Sigma, j)$  is a closed connected Riemann surface of genus  $g$ ,  $\Theta \subset \Sigma$  is an ordered set of distinct points (the **marked points**), and  $u : (\Sigma, j) \rightarrow (M, J)$  is a  $J$ -holomorphic map satisfying  $[u] := u_*[\Sigma] = A$ , with equivalence defined by  $(\Sigma, j, \Theta, u) \sim (\Sigma', \psi^*j, \psi^{-1}(\Theta), u \circ \psi)$  for diffeomorphisms  $\psi : \Sigma' \rightarrow \Sigma$ . The **Gromov compactification** of  $\mathcal{M}_{g,m}(A, J)$  is the space  $\overline{\mathcal{M}}_{g,m}(A, J)$  (of equivalence classes of) **stable nodal curves**  $(S, j, \Theta, \Delta, u)$ , where now  $S$  may be disconnected, and the original data are augmented by an unordered set of distinct points in  $S \setminus \Theta$ , arranged into unordered pairs

$$\Delta = \{\{\hat{z}_1, \check{z}_1\}, \dots, \{\hat{z}_r, \check{z}_r\}\},$$

such that  $u(\hat{z}_i) = u(\check{z}_i)$  for each  $i = 1, \dots, r$ . We call the pairs  $\{\hat{z}_i, \check{z}_i\}$  **nodes**, and each individual  $\hat{z}_i$  or  $\check{z}_i \in S$  a **nodal point**. The curves in  $\overline{\mathcal{M}}_{g,m}(A, J)$  are required to have **arithmetic genus**  $g$ , which means that the surface obtained from  $S$  by performing connected sums at all matched pairs of nodal points is a closed connected surface of genus  $g$ . The stability condition requires that any component of  $S \setminus (\Theta \cup \Delta)$  on which  $u$  is constant should have negative Euler characteristic. With this condition,  $\overline{\mathcal{M}}_{g,m}(A, J)$  can be given a natural topology as a metrizable Hausdorff space, and it is compact whenever  $J$  is tamed by a symplectic form. A definition of the topology may be found e.g. in [BEH<sup>+</sup>03]; for sequences in  $\mathcal{M}_{g,m}(A, J)$ , it amounts to the notion of  $C^\infty$ -convergence for  $j$  and  $u$  after a choice of parametrization for which all domains and marked point sets are identified. Curves  $[(S, j, \Theta, \Delta, u)] \in \overline{\mathcal{M}}_{g,m}(A, J)$  with  $\Delta = \emptyset$  can equivalently be regarded as elements of  $\mathcal{M}_{g,m}(A, J)$ , and are thus called **smooth** curves to distinguish them from nodal curves.

*Remark 2.1.* In this paper, the word “curve” always means “smooth curve” (i.e. without nodes) unless the word “nodal” is explicitly included. Similarly, all dimensions and Fredholm indices in this paper are *real* (not complex) unless otherwise specified. This usage differs somewhat from the algebraic geometry literature.

When there is no danger of confusion, we shall sometimes abuse notation by denoting equivalence classes  $[(\Sigma, j, \Theta, u)] \in \mathcal{M}_{g,m}(A, J)$  or  $[(S, j, \Theta, \Delta, u)] \in \overline{\mathcal{M}}_{g,m}(A, J)$  simply by  $u \in \mathcal{M}_{g,m}(A, J)$  or  $u \in \overline{\mathcal{M}}_{g,m}(A, J)$  respectively, and we will refer to the restriction of a nodal curve  $[(S, j, \Theta, \Delta, u)]$  to any connected component of its domain  $S$  as a **smooth component** of  $u$ . We shall also abbreviate

$$\mathcal{M}_g(A, J) := \mathcal{M}_{g,0}(A, J), \quad \text{and} \quad \overline{\mathcal{M}}_g(A, J) := \overline{\mathcal{M}}_{g,0}(A, J).$$



Recall that  $\mathcal{M}_g(A, J)$  has **virtual dimension** equal to the index of any curve  $u \in \mathcal{M}_g(A, J)$ , while the virtual dimension of the moduli space with marked points is

$$\text{vir-dim } \mathcal{M}_{g,m}(A, J) = \text{vir-dim } \mathcal{M}_g(A, J) + 2m.$$

The multiply covered curves form a distinguished closed subset of  $\overline{\mathcal{M}}_g(A, J)$ . Given any  $u \in \mathcal{M}_g(A, J)$  with domain  $(\Sigma, j)$ , and integers  $h \geq 0$ ,  $d \geq 1$ , define the space of stable **nodal  $d$ -fold covers** of  $u$ ,

$$\overline{\mathcal{M}}_h(d; u) = \{[(S, \tilde{j}, \Delta, u \circ \varphi)] \in \overline{\mathcal{M}}_h(dA, J) \mid [(S, \tilde{j}, \Delta, \varphi)] \in \overline{\mathcal{M}}_h(d[\Sigma], j)\},$$

so in particular, each smooth component  $\tilde{u}_i$  of  $\tilde{u} \in \overline{\mathcal{M}}_h(d; u)$  belongs to a space  $\mathcal{M}_{g_i}(d_i; u)$  of smooth branched covers  $u \circ \varphi_i$  of some degree  $d_i \geq 0$ , such that  $\sum_i d_i = d$ . Note that  $\overline{\mathcal{M}}_h(d; u)$  may in general be strictly larger than the closure of  $\mathcal{M}_h(d; u)$  in the Gromov topology—to cite one well-known example, the space  $\mathcal{M}_1([S^2], i)$  of smooth degree 1 holomorphic tori in  $(S^2, i)$  is empty, but  $\overline{\mathcal{M}}_1([S^2], i)$  contains a nodal curve with a constant component of genus 1.

In this language, the main consequence of Proposition 1.3 can be stated as follows.

**Corollary 2.2.** *Suppose  $(M, J)$  is an almost complex manifold and  $u \in \mathcal{M}_g(A, J)$  is a super-rigid curve in  $M$ . Then for every  $h \geq 0$  and  $d \geq 1$ ,  $\overline{\mathcal{M}}_h(d; u)$  is an open and closed subset of  $\overline{\mathcal{M}}_h(dA, J)$ .*

Recall next that every  $J$ -holomorphic curve  $u : (\Sigma, j) \rightarrow (M, J)$  gives rise to a **linearized Cauchy-Riemann operator**

$$\mathbf{D}_u : \Gamma(u^*TM) \rightarrow \Omega^{0,1}(\Sigma, u^*TM),$$

i.e. the linearization at  $u$  of the nonlinear Cauchy-Riemann operator  $\bar{\partial}_J(u) := Tu + J \circ Tu \circ j \in \Omega^{0,1}(\Sigma, u^*TM)$ , whose zero-set is the space of all  $J$ -holomorphic maps. The operator  $\mathbf{D}_u$  takes vector fields along  $u$  to  $(0, 1)$ -forms valued in the complex vector bundle  $(u^*TM, J)$ , and can be written explicitly as

$$\mathbf{D}_u \eta = \nabla \eta + J(u) \circ \nabla \eta \circ j + (\nabla_\eta J) \circ Tu \circ j$$

for any choice of symmetric connection  $\nabla$  (cf. [Wen, §2.4]). Recall moreover that whenever  $u$  is nonconstant, its critical points are isolated and one can find a smooth splitting of complex vector bundles

$$(2.1) \quad u^*TM = T_u \oplus N_u$$

such that  $T_u$  matches the image of  $du$  at regular points; see e.g. [Wen10, §3.3] for details. In many cases of interest in this paper,  $u$  will be a cover of an immersed  $J$ -holomorphic curve  $v$ , so  $N_u$  is then simply the pullback of the normal bundle of  $v$  via the cover. We define the **normal Cauchy-Riemann operator** at  $u$  as the restriction of  $\mathbf{D}_u$  to sections of  $N_u$ , composed with the projection  $\pi_N : u^*TM \rightarrow N_u$ , hence

$$\mathbf{D}_u^N = \pi_N \circ \mathbf{D}_u|_{\Gamma(N_u)} : \Gamma(N_u) \rightarrow \Omega^{0,1}(\Sigma, N_u).$$

In general, a neighborhood of any element in  $\mathcal{M}_{g,m}(A, J)$  can be identified with the zero-set of a smooth Fredholm section of a Banach space bundle, modulo a finite group action if there are nontrivial automorphisms. We say that  $u \in \mathcal{M}_g(A, J)$  is **Fredholm regular** whenever it is a transverse intersection of this section with the zero-section. Note

that whenever this condition holds, it automatically also holds after adding any finite collection of marked points and viewing  $u$  as an element of  $\mathcal{M}_{g,m}(A, J)$ . The implicit function theorem gives the open set of regular curves in  $\mathcal{M}_{g,m}(A, J)$  the structure of a smooth orbifold with dimension equal to its virtual dimension, and local isotropy groups determined by the automorphism groups of the curves—in particular, the set of regular simple curves forms a manifold, though orbifold singularities can appear when multiple covers are included. The following convenient repackaging of the regularity condition comes from [Wen10, Corollary 3.13].

**Proposition 2.3.** *A closed and connected  $J$ -holomorphic curve  $u : (\Sigma, j) \rightarrow (M, J)$  is Fredholm regular if and only if its normal Cauchy-Riemann operator  $\mathbf{D}_u^N : W^{k,p}(N_u) \rightarrow W^{k-1,p}(\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, N_u))$  is surjective for some (and therefore all)  $k \in \mathbb{N}$  and  $p \in (1, \infty)$ .  $\square$*

**Definition 2.4.** A closed, connected, simple  $J$ -holomorphic curve  $u : (\Sigma, j) \rightarrow (M, J)$  is called **super-rigid** if it satisfies the following:

- (1)  $\text{ind}(u) = 0$ ;
- (2)  $u : \Sigma \rightarrow M$  is an immersion;
- (3) For all closed connected Riemann surfaces  $(\tilde{\Sigma}, \tilde{j})$  and holomorphic maps  $\varphi : (\tilde{\Sigma}, \tilde{j}) \rightarrow (\Sigma, j)$  of positive degree, the curve  $\tilde{u} := u \circ \varphi : (\tilde{\Sigma}, \tilde{j}) \rightarrow (M, J)$  admits no nontrivial solutions to the normal linearized equation  $\mathbf{D}_{\tilde{u}}^N \eta = 0$ .

**2.2. A stratification theorem.** We now explain in precise terms the stratification result that underlies the main theorems of §1.1.

Suppose  $v : (\Sigma, j) \rightarrow (M, J)$  is a simple curve with genus  $g \geq 0$  and  $\varphi : (\Sigma', j') \rightarrow (\Sigma, j)$  is a branched cover with degree  $d > 1$ , giving rise to the multiply covered curve  $u = v \circ \varphi : (\Sigma', j') \rightarrow (M, J)$  of genus  $h \geq 0$ . For the sake of intuition, we begin under the assumption  $d = 2$ . Then the automorphism group

$$\text{Aut}(u) = \text{Aut}(\varphi) := \left\{ \psi : (\Sigma', j') \xrightarrow{\cong} (\Sigma', j') \mid \varphi = \varphi \circ \psi \right\}$$

contains a unique nontrivial element  $\psi$ , and the space of sections  $\Gamma(N_u)$  has a natural splitting

$$\Gamma(N_u) = \Gamma_+(N_u) \oplus \Gamma_-(N_u)$$

where  $\Gamma_{\pm}(N_u) := \{\eta \in \Gamma(N_u) \mid \eta = \pm \eta \circ \psi\}$ . Splitting  $\Omega^{0,1}(\Sigma', N_u) = \Gamma(\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma', N_u))$  and suitable Sobolev spaces of sections on these two bundles in the same way, one obtains a splitting of the normal Cauchy-Riemann operator

$$(2.2) \quad \mathbf{D}_u^N = \mathbf{D}_{u,+}^N \oplus \mathbf{D}_{u,-}^N$$

into two operators  $\mathbf{D}_{u,\pm}^N : W_{\pm}^{k,p}(N_u) \rightarrow W_{\pm}^{k-1,p}(\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma', N_u))$ . It is not hard to see that  $\mathbf{D}_{u,+}^N$  is in some sense equivalent to  $\mathbf{D}_v^N$ , as its domain and target both consist of sections that are pullbacks via  $\varphi$  of sections over  $\Sigma$ . The standard transversality theory for simple curves thus shows that  $\mathbf{D}_{u,+}^N$  can be assumed surjective (and also injective if  $v$  is immersed with index 0) if  $J$  is chosen generically. We will see that the problem of proving surjectivity or injectivity for  $\mathbf{D}_u^N$  becomes tractable when viewed as two independent problems for the operators  $\mathbf{D}_{u,+}^N$  and  $\mathbf{D}_{u,-}^N$ .

In order to generalize this discussion beyond the degree 2 case, it helps to adopt an alternative perspective based on representation theory. Let  $\Theta \subset \Sigma$  denote a finite subset that contains all critical values of  $\varphi$ , and set

$$(2.3) \quad \Theta' := \varphi^{-1}(\Theta), \quad \dot{\Sigma} := \Sigma \setminus \Theta, \quad \dot{\Sigma}' := \Sigma' \setminus \Theta',$$

so that  $\dot{\Sigma}' \xrightarrow{\varphi} \dot{\Sigma}$  is a smooth covering map with  $G := \text{Aut}(\varphi) \cong \mathbb{Z}_2$  as its group of deck transformations. Define

$$\rho : G \rightarrow S_2 : g \mapsto \rho_g$$

as the isomorphism to the symmetric group on  $\{1, 2\}$ . We can then identify the covering map  $\dot{\Sigma}' \xrightarrow{\varphi} \dot{\Sigma}$  with

$$\left( \dot{\Sigma}' \times \{1, 2\} \right) / G \rightarrow \dot{\Sigma} : [(z, i)] \mapsto \varphi(z),$$

where  $G$  acts on  $\dot{\Sigma}'$  by deck transformations and on  $\{1, 2\}$  via  $\rho$ . Now if  $(e_1, e_2)$  denotes the standard basis of  $\mathbb{R}^2$ , then  $\rho$  also gives rise to a real permutation representation

$$\rho : G \rightarrow \text{GL}(2, \mathbb{R}), \quad \rho(g)e_i := e_{\rho_g(i)},$$

and a corresponding real vector bundle  $V^\rho \rightarrow \dot{\Sigma}$  defined as the  $\mathbb{Z}_2$ -quotient of a trivial bundle over  $\dot{\Sigma}'$ ,

$$V^\rho := \left( \dot{\Sigma}' \times \mathbb{R}^2 \right) / G.$$

The space of sections of the twisted normal bundle

$$N_v^\rho := N_v \otimes_{\mathbb{R}} V^\rho \rightarrow \dot{\Sigma}$$

then has a natural identification with the space of sections of  $N_u = \varphi^* N_v$ : indeed, we can represent sections of  $N_v^\rho$  as  $\mathbb{Z}_2$ -equivariant sections  $\eta = \sum_{i=1}^2 \eta^i \otimes e_i$  of  $\varphi^* N_v \otimes_{\mathbb{R}} \mathbb{R}^2$ , which satisfy the relation  $\eta^i \circ \psi = \eta^{\rho_\psi(i)}$ , thus a corresponding section  $\hat{\eta} \in \Gamma(\varphi^* N_v)$  can be defined under the identification of  $\dot{\Sigma}$  with  $(\dot{\Sigma} \times \{1, 2\})/G$  by

$$\hat{\eta}([(z, i)]) = \eta^i(z).$$

Under this identification,  $\mathbf{D}_u^N$  becomes a Cauchy-Riemann type operator on the twisted bundle  $N_v^\rho$ , defined locally by  $\mathbf{D}_u^N(\eta \otimes v) = (\mathbf{D}_v^N \eta) \otimes v$  whenever  $v$  is a local section of  $V^\rho$  that has a constant lift to the trivial bundle  $\dot{\Sigma}' \times \mathbb{R}^2$ .

The above construction appears cumbersome at first glance, but it has the following advantage: the decomposition  $\Gamma(N_u) = \Gamma_+(N_u) \oplus \Gamma_-(N_u)$  now corresponds to a splitting of the twisted bundle  $N_v^\rho$  into subbundles

$$N_v^\rho = N_v^{\theta_+} \oplus N_v^{\theta_-} := (N_v \otimes_{\mathbb{R}} V^{\theta_+}) \oplus (N_v \otimes_{\mathbb{R}} V^{\theta_-})$$

where  $V^{\theta_\pm} := (\dot{\Sigma}' \times W_\pm)/G$  are defined in terms of the natural splitting of  $\mathbb{R}^2 = W_+ \oplus W_-$  into irreducible  $G$ -invariant subspaces

$$W_\pm = \mathbb{R} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \subset \mathbb{R}^2.$$

This is the simplest nontrivial example of what turns out to be a general principle: splittings of Cauchy-Riemann operators for multiply covered curves arise from decompositions of permutation representations into irreducible summands. To turn  $\rho = \theta_+ \oplus \theta_-$  into a splitting of Cauchy-Riemann operators, we still have a small analytical issue to cope with

since the bundles  $N_v^{\theta^\pm}$  are defined over  $\dot{\Sigma}$  and do not both extend over the punctures. In place of (2.2), we therefore obtain a splitting

$$\dot{\mathbf{D}}_u^N = \dot{\mathbf{D}}_{u,\theta_+}^N \oplus \dot{\mathbf{D}}_{u,\theta_-}^N,$$

where the dots over the operators indicate that we are restricting them to the punctured domain  $\dot{\Sigma}'$ . We will see in §3.2 how to define suitable weighted Sobolev spaces over  $\dot{\Sigma}$  and  $\dot{\Sigma}'$  so that the punctured operators have the same indices, kernels and cokernels as their unpunctured counterparts.

We return now to the general case and continue using the notation  $\dot{\Sigma}' \xrightarrow{\varphi} \dot{\Sigma}$  for the covering map obtained by deleting some finite subsets that include the critical values and their preimages. Recall that  $\varphi$  is called **regular** if  $|\text{Aut}(\varphi)| = \deg(\varphi) = d$ . This condition was secretly important in the above discussion of the  $d = 2$  case, as the definition of the twisted bundle  $N_v^P$  required identifying  $\dot{\Sigma}$  with the quotient of  $\dot{\Sigma}'$  by deck transformations. In general,  $\text{Aut}(\varphi)$  may be trivial, but we can use some notions from elementary covering space theory to get around this.

**Definition 2.5.** The **generalized automorphism group** of a  $d$ -fold branched cover  $\varphi : \Sigma' \rightarrow \Sigma$  is the quotient  $G := \pi_1(\dot{\Sigma})/H$ , where  $H$  is the normal core of  $\varphi_*(\pi_1(\dot{\Sigma}'))$ , and  $\dot{\Sigma}$  and  $\dot{\Sigma}'$  are defined by (2.3) with  $\Theta$  as the set of critical values of  $\varphi$ .

**Definition 2.6.** A **regular presentation** of the holomorphic  $d$ -fold branched cover  $\varphi : (\Sigma', j') \rightarrow (\Sigma, j)$  is a tuple  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$  consisting of:

- A finite subset  $\Theta \subset \Sigma$  containing the critical values of  $\varphi$  and defining the punctured surfaces  $\dot{\Sigma}$  and  $\dot{\Sigma}'$  via (2.3);
- A connected surface  $\dot{\Sigma}''$  and regular covering map  $\pi : \dot{\Sigma}'' \rightarrow \dot{\Sigma}$  with finite automorphism group  $G := \text{Aut}(\pi)$ ;
- A set  $I$  with  $d$  elements;
- A transitive action of  $G$  on  $I$ , defined via a homomorphism  $\rho : G \rightarrow S(I)$  from  $G$  to the symmetric group on  $I$ ;
- A diffeomorphism  $f : \dot{\Sigma}' \rightarrow (\dot{\Sigma}'' \times I)/G$ , where  $G$  acts on  $\dot{\Sigma}''$  by deck transformations and on  $I$  via  $\rho$ , such that  $\varphi \circ f^{-1}$  takes the form

$$\left( \dot{\Sigma}'' \times I \right) / G \rightarrow \dot{\Sigma} : [(z, i)] \mapsto \pi(z).$$

We say that  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$  is **minimal** if  $\Theta \subset \Sigma$  is the set of critical values of  $\varphi$  and  $\rho : G \rightarrow S(I)$  is injective. Two regular presentations  $(\Theta_j, \dot{\Sigma}_j'', \pi_j, G_j, \rho_j, I_j, f_j)$  of  $\varphi : \Sigma' \rightarrow \Sigma$  for  $j = 1, 2$  are **isomorphic** if  $\Theta_1 = \Theta_2$  and there exists a diffeomorphism  $\Psi : \dot{\Sigma}_1'' \rightarrow \dot{\Sigma}_2''$ , a bijection  $\beta : I_1 \rightarrow I_2$ , and a group isomorphism  $\Phi : G_1 \rightarrow G_2$  such that:

- (1)  $\pi_2 \circ \Psi = \pi_1$  and for all  $g \in G_1$ ,  $\Psi \circ g = \Phi(g) \circ \Psi$ ;
- (2) For all  $g \in G_1$ ,  $\beta \circ \rho_1(g) = \rho_2(\Phi(g)) \circ \beta$ ;
- (3)  $f_2 \circ f_1^{-1}$  takes the form

$$\left( \dot{\Sigma}_1'' \times I_1 \right) / G_1 \rightarrow \left( \dot{\Sigma}_2'' \times I_2 \right) / G_2 : [(z, i)] \mapsto [(\Psi(z), \beta(i))].$$

We will show in §3.1 that the regular cover  $\pi : \dot{\Sigma}'' \rightarrow \dot{\Sigma}$  in any regular presentation can be extended to a holomorphic branched cover of closed connected Riemann surfaces

$(\Sigma'', j'') \rightarrow (\Sigma, j)$  such that  $\dot{\Sigma}'' = \Sigma'' \setminus \pi^{-1}(\Theta)$ . Observe that if  $i \in I$  and  $G_i \subset G$  denotes the stabilizer of  $i$  under the  $G$ -action defined by  $\rho$ , then

$$\dot{\Sigma}''/G_i \rightarrow (\dot{\Sigma}'' \times I) / G : [z] \mapsto [(z, i)]$$

is a diffeomorphism identifying  $\varphi \circ f^{-1}$  with the natural projection  $\dot{\Sigma}''/G_i \rightarrow \dot{\Sigma}''/G = \dot{\Sigma}$ . Thus one can associate to any regular presentation a (non-unique) factorization of  $\pi : \dot{\Sigma}'' \rightarrow \dot{\Sigma}$  by covering maps  $\dot{\Sigma}'' \rightarrow \dot{\Sigma}' \xrightarrow{\varphi} \dot{\Sigma}$ , which extends over the punctures to a factorization of  $\pi : (\Sigma'', j'') \rightarrow (\Sigma, j)$  by holomorphic branched covers

$$(\Sigma'', j'') \rightarrow (\Sigma', j') \xrightarrow{\varphi} (\Sigma, j).$$

We will also show in Lemma 3.2 that  $\varphi : \Sigma' \rightarrow \Sigma$  always admits a unique isomorphism class of minimal regular presentations  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ , for which  $G$  is isomorphic to the generalized automorphism group of  $\varphi$ , and in this case  $\pi : \dot{\Sigma}'' \rightarrow \dot{\Sigma}$  is isomorphic to  $\varphi : \Sigma' \rightarrow \Sigma$  whenever the latter happens to be already regular.

Given a choice of regular presentation  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ , the discussion of the degree 2 case can be generalized as follows. The transitive action  $\rho : G \rightarrow S(I)$  induces a permutation representation  $\rho : G \rightarrow \text{Aut}_{\mathbb{R}}(\mathbb{R}^I)$  on the real vector space  $\mathbb{R}^I$  with basis labeled by the elements of  $I$ , and a twisted bundle  $N_v^{\rho} = N_v \otimes_{\mathbb{R}} V^{\rho} \rightarrow \dot{\Sigma}$ , where

$$V^{\rho} := (\dot{\Sigma}'' \times \mathbb{R}^I)/G,$$

with a natural isomorphism

$$\Gamma(N_v^{\rho}) = \Gamma(\varphi^* N_v|_{\dot{\Sigma}'}) = \Gamma(N_u|_{\dot{\Sigma}'})$$

that identifies  $\mathbf{D}_u^N$  with a Cauchy-Riemann operator  $\dot{\mathbf{D}}_{u, \rho}^N$  on suitable exponentially weighted Sobolev spaces of sections of  $N_v^{\rho}$ . Any finite-dimensional real representation  $\theta : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$  similarly gives rise to a twisted bundle  $N_v^{\theta} = N_v \otimes_{\mathbb{R}} V^{\theta} \rightarrow \dot{\Sigma}$ , with  $V^{\theta} := (\dot{\Sigma}'' \times W)/G$ , and a twisted Cauchy-Riemann operator  $\dot{\mathbf{D}}_{u, \theta}^N$  which (up to conjugacy) depends only on  $\dot{\mathbf{D}}_v^N$  and the isomorphism classes of the regular presentation and the representation  $\theta$ . Now any representation-theoretic decomposition  $\rho = \theta_1^{\oplus k_1} \oplus \dots \oplus \theta_N^{\oplus k_N}$  induces a splitting of the punctured Cauchy-Riemann operator

$$(2.4) \quad \mathbf{D}_u^N \cong \dot{\mathbf{D}}_{u, \rho}^N = (\dot{\mathbf{D}}_{u, \theta_1}^N)^{\oplus k_1} \oplus \dots \oplus (\dot{\mathbf{D}}_{u, \theta_N}^N)^{\oplus k_N},$$

with the following useful property:

**Lemma 2.7.** *The normal Cauchy-Riemann operator  $\mathbf{D}_u^N$  for a multiple cover is surjective or injective if and only if the same holds for all of the summands  $\dot{\mathbf{D}}_{u, \theta_j}^N$  in (2.4).*

*Remark 2.8.* We will see below that the splitting (2.4) can be arranged to vary smoothly as  $v$  and  $\varphi$  move about in their respective moduli spaces, so the indices of the summands  $\dot{\mathbf{D}}_{u, \theta_j}^N$  are constant. This immediately gives rise to “no-go” results about transversality and super-rigidity: the former is impossible unless all the  $\dot{\mathbf{D}}_{u, \theta_j}^N$  have nonnegative index, and the latter unless they have nonpositive index. Conversely, whenever either of these index conditions holds for all summands given by irreducible representations, Theorem D below will imply that the desired transversality or super-rigidity result holds for all pairs  $(v, \varphi)$  lying in some open and dense subset. This is the main idea behind Theorem C, and

it similarly provides a necessary and sufficient criterion for the feasibility of obstruction bundle arguments.

It should be emphasized that the representations of  $G$  in this discussion are *real*, not complex. We will need to use the standard fact (see §3.3) that for any finite group  $G$ , real irreducible representations  $\theta : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$  come in three types, characterized via the space  $\text{End}_G(W)$  of  $G$ -equivariant real-linear maps  $W \rightarrow W$ :

- **Real type:**  $\text{End}_G(W) \cong \mathbb{R}$
- **Complex type:**  $\text{End}_G(W) \cong \mathbb{C}$
- **Quaternionic type:**  $\text{End}_G(W) \cong \mathbb{H}$

In order to discuss what happens to (2.4) as  $v$  and  $\varphi$  move in their respective moduli spaces, we observe that the construction depends quite heavily on the branching structure of  $\varphi : \Sigma' \rightarrow \Sigma$ , i.e. the number of punctures  $\Theta' \subset \Sigma'$  and the topological behavior of  $\varphi$  in their vicinity. This necessitates decomposing the space of all degree  $d$  branched covers into strata

$$\bigcup_{h \geq 0} \mathcal{M}_h(d[\Sigma], j) = \bigcup_{\mathbf{b}} \mathcal{M}_{\mathbf{b}}^d(j)$$

labeled by their so-called *branching data*  $\mathbf{b}$ . Choose an integer  $r \geq 0$ , and associate to each of the numbers  $i = 1, \dots, r$  a nonempty finite ordered set of natural numbers

$$\mathbf{b}_i = (b_i^1, \dots, b_i^{q_i})$$

such that

$$b_i^1 + \dots + b_i^{q_i} = d$$

and at least one of the numbers  $b_i^1, \dots, b_i^{q_i}$  is strictly greater than 1. We denote the totality of this data by  $\mathbf{b} = (\mathbf{b}_1, \dots, \mathbf{b}_r)$  and call it **branching data of degree  $d$  with  $r$  critical values**. Given this, let  $\widetilde{\mathcal{M}}_{\mathbf{b}}^d(j)$  denote the moduli space of all closed and connected unparametrized  $j$ -holomorphic curves  $\varphi$  in  $(\Sigma, j)$  of degree  $d$  with  $q_1 + \dots + q_r$  marked points

$$\zeta_1^1, \dots, \zeta_1^{q_1}, \zeta_2^1, \dots, \zeta_2^{q_2}, \dots, \zeta_r^1, \dots, \zeta_r^{q_r}$$

such that

- (1) there are distinct points  $w_1, \dots, w_r \in \Sigma$  such that  $\varphi^{-1}(w_i) = \{\zeta_i^1, \dots, \zeta_i^{q_i}\}$  for each  $i = 1, \dots, r$ ;
- (2) for each  $i = 1, \dots, r$  and  $j = 1, \dots, q_i$ ,  $\varphi$  is locally  $b_i^j$ -to-1 near  $\zeta_i^j$ ; and
- (3)  $\varphi$  has no critical points outside of the marked points.

Note that we do not require *every* marked point of  $\varphi$  to be a critical point, but we are assuming  $\{w_1, \dots, w_r\}$  is the set of critical values, whose preimages are marked points and may include both critical and regular points. For any  $\varphi \in \widetilde{\mathcal{M}}_{\mathbf{b}}^d(j)$ , we have

$$Z(d\varphi) = \sum_{i=1}^r \sum_{j=1}^{q_i} (b_i^j - 1),$$

thus  $d$  and  $\mathbf{b}$  determine the genus  $h$  of  $\varphi$  via the Riemann-Hurwitz formula, and we shall denote by

$$\mathcal{M}_{\mathbf{b}}^d(j) \subset \mathcal{M}_h(d[\Sigma], j)$$



the image of the natural map  $\widetilde{\mathcal{M}}_{\mathbf{b}}^d(j) \hookrightarrow \mathcal{M}_h(d[\Sigma], j)$  defined by forgetting the marked points. Note that in some cases, the Riemann-Hurwitz calculation may produce a negative genus, which just means that  $\mathcal{M}_{\mathbf{b}}^d(j)$  is empty. If  $\mathbf{b}$  is empty, i.e.  $r = 0$ , it means every  $\varphi \in \mathcal{M}_{\mathbf{b}}^d(j)$  is unbranched.

It is a classical fact that  $\mathcal{M}_{\mathbf{b}}^d(j)$  is a smooth manifold of real dimension  $2r$ , as it can be parametrized locally by the positions of the critical values  $w_1, \dots, w_r \in \Sigma$ . Moreover, it depends smoothly on  $j$  in the sense that if  $P$  is any smooth finite-dimensional family of complex structures on  $\Sigma$ , then

$$\bigcup_{j \in P} \mathcal{M}_{\mathbf{b}}^d(j) \rightarrow P$$

defines a smooth fiber bundle. We will show in §3.1 that regular presentations of  $\varphi : \Sigma' \rightarrow \Sigma$  can also be arranged to vary smoothly as  $\varphi$  varies with fixed branching data.

We are now nearly ready to state the main technical result of the paper. Given integers  $m \geq 0$  and  $\ell_1, \dots, \ell_m \geq 1$ , let

$$\mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m) \subset \mathcal{M}_{g,m}(A, J)$$

denote the subset consisting of curves that have critical points of critical order  $\ell_i$  at the  $i$ th marked point for  $i = 1, \dots, m$  and are immersed everywhere else. As explained in Appendix A, the simple curves in this space form a smooth submanifold for generic  $J$ , with codimension  $2n \sum_i \ell_i$  in  $\mathcal{M}_{g,m}(A, J)$ . Given a finite group  $G$ , an integer  $d \in \mathbb{N}$  and branching data  $\mathbf{b}$  of degree  $d$  with  $r \geq 0$  critical values, define

$$\mathcal{M}_{\mathbf{b},G}^d(\mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m)) \subset \mathcal{M}_h(dA, J)$$

so be the set of all curves with representatives of the form  $u = v \circ \varphi : (\Sigma', j') \rightarrow (M, J)$  where  $\varphi : (\Sigma', j') \rightarrow (\Sigma, j)$  represents an element in  $\mathcal{M}_{\mathbf{b}}^d(j)$  with generalized automorphism group isomorphic to  $G$ , and  $v : (\Sigma, j) \rightarrow (M, J)$  is a simple curve with  $m$  marked points representing an element of  $\mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m)$  that intersects the subset  $\mathcal{U} \subset M$ . Note that if  $J$  is generic on  $\mathcal{U}$ , then standard results give  $\mathcal{M}_{\mathbf{b},G}^d(\mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m))$  the structure of a smooth manifold with

$$\dim \mathcal{M}_{\mathbf{b},G}^d(\mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m)) = 2r + (n-3)(2-2g) + 2c_1(A) - 2 \sum_{i=1}^m (n\ell_i - 1).$$

Now fix a complete list of pairwise non-isomorphic irreducible real representations

$$\theta_i : G \rightarrow \text{Aut}_{\mathbb{R}}(W_i), \quad i = 1, \dots, N,$$

and for any  $N$ -tuples of nonnegative integers  $\mathbf{k} = (k_1, \dots, k_N)$  and  $\mathbf{c} = (c_1, \dots, c_N)$ , let

$$\mathcal{M}_{\mathbf{b},G}^d(\mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m); \mathbf{k}, \mathbf{c}) \subset \mathcal{M}_{\mathbf{b},G}^d(\mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m))$$

denote the subset determined by the conditions

$$\dim \ker \dot{\mathbf{D}}_{u, \theta_i}^N = k_i \quad \text{and} \quad \dim \text{coker } \dot{\mathbf{D}}_{u, \theta_i}^N = c_i \quad \text{for all } i = 1, \dots, N,$$

where the twisted operators  $\dot{\mathbf{D}}_{u, \theta_i}^N$  are defined with respect to the unique isomorphism class of minimal regular presentations.

**Theorem D** (stratification). *There exists a Baire subset  $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$  such that the following holds for all  $J \in \mathcal{J}_{\text{reg}}$ . For all choices of a finite group  $G$ , branching data  $\mathbf{b}$ , integers  $g, m \geq 0$ ,  $\ell_1, \dots, \ell_m \geq 1$  and a homology class  $A \in H_2(M)$ , the subset*

$$\mathcal{M}_{\mathbf{b}, G}^d(\mathcal{M}_{g, m}(A, J; \ell_1, \dots, \ell_m); \mathbf{k}, \mathbf{c}) \subset \mathcal{M}_{\mathbf{b}, G}^d(\mathcal{M}_{g, m}(A, J; \ell_1, \dots, \ell_m))$$

*is a smooth submanifold with*

$$\text{codim } \mathcal{M}_{\mathbf{b}, G}^d(\mathcal{M}_{g, m}(A, J; \ell_1, \dots, \ell_m); \mathbf{k}, \mathbf{c}) = \sum_{i=1}^N t_i k_i c_i,$$

*where for each  $i = 1, \dots, N$ ,  $t_i = 1, 2, 4$  according to whether the representation  $\theta_i$  is of real, complex or quaternionic type respectively.*

*Remark 2.9.* The statement above is specifically geared toward the applications treated in this paper, but for different purposes one could formulate various other versions, e.g. one could add more marked points to  $\mathcal{M}_{g, m}(A, J; \ell_1, \dots, \ell_m)$  and impose intersection constraints on them, or one could consider generic finite-dimensional families  $\{J_s\}_{s \in P}$  of almost complex structures and thus replace  $\mathcal{M}_{g, m}(A, J; \ell_1, \dots, \ell_m)$  with a parametric moduli space of pairs  $(u, s)$  where  $s \in P$  and  $u$  is  $J_s$ -holomorphic. Either would require no serious modifications to the proof, other than more cumbersome notation.

We need two further ingredients in order to turn Theorem D into a powerful enough tool for proving the theorems of §1.1. The first is an index calculation for the twisted operators  $\dot{\mathbf{D}}_{u, \theta}^N$ . The precise result is stated and proved in §4, but for the main applications we only need the following estimate, which is a corollary:

**Lemma 2.10.** *Given a  $J$ -holomorphic curve  $v : (\Sigma, j) \rightarrow (M, J)$  with normal Cauchy-Riemann operator  $\mathbf{D}_v^N$ , a  $d$ -fold branched cover  $\varphi : (\Sigma', j') \rightarrow (\Sigma, j)$  with  $r \geq 0$  critical values, a regular presentation  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$  for  $\varphi$  and a real representation  $\theta : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$ , the resulting twisted Cauchy-Riemann operator  $\dot{\mathbf{D}}_{u, \theta}^N$  for  $u = v \circ \varphi$  satisfies*

$$\dim W \cdot [\text{ind}(\mathbf{D}_v^N) - (n-1)r] \leq \text{ind}(\dot{\mathbf{D}}_{u, \theta}^N) \leq \dim W \cdot \text{ind}(\mathbf{D}_v^N).$$

*Moreover, if the regular presentation is minimal and  $\theta$  is faithful, then the second estimate can be improved to*

$$\text{ind}(\dot{\mathbf{D}}_{u, \theta}^N) \leq \dim W \cdot \text{ind}(\mathbf{D}_v^N) - (n-1)r,$$

*and it is strict unless all branch points of  $\varphi$  have branching order 2.*

For the proof of super-rigidity, we will need the next result as a means of improving the upper bound in Lemma 2.10 for representations that are not faithful.

**Lemma 2.11** (see §3.4.3). *Under the assumptions of Lemma 2.10, suppose the regular presentation is minimal, and the splitting (2.4) of  $\mathbf{D}_u^N$  includes a summand  $\dot{\mathbf{D}}_{u, \theta}^N$  for which the representation  $\theta : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$  is not faithful. Then  $\varphi : (\Sigma', j') \rightarrow (\Sigma, j)$  admits a factorization by holomorphic branched covers*

$$(\Sigma', j') \rightarrow (\Sigma'_0, j'_0) \xrightarrow{\varphi_0} (\Sigma, j)$$

with  $\deg(\varphi_0) < d$ , and  $\dot{\mathbf{D}}_{u,\theta}^N$  is conjugate to an operator  $\dot{\mathbf{D}}_{u_0,\theta_0}^N$  defined with respect to a regular presentation  $(\Theta, \dot{\Sigma}_0'', \pi_0, G_0, \rho_0, I_0, f_0)$  for  $\varphi_0$ , where  $u_0 := v \circ \varphi_0 : (\Sigma'_0, j'_0) \rightarrow (M, J)$ ,  $G_0 := G / \ker \theta$ , and

$$\theta_0 : G / \ker \theta \rightarrow \text{Aut}_{\mathbb{R}}(W)$$

is the faithful representation of  $G_0$  determined by  $\theta$ . Moreover,  $\mathbf{D}_{u_0}^N$  also admits a splitting in the form (2.4) which has  $\dot{\mathbf{D}}_{u_0,\theta_0}^N$  as a summand.

**2.3. Proof of the main theorems modulo stratification.** Let us now take the results of the previous section as black boxes and prove the main theorems from §1.1.

*Proof of Theorem A in dimension greater than four.* We argue by induction on the degrees  $d \in \mathbb{N}$  of branched covers. For  $d = 1$ , we only need to know that generic perturbations of  $J$  suffice to make all simple index 0 curves through  $\mathcal{U}$  regular and immersed; this is standard (see Appendix A for the immersion property). Thus for  $d \geq 2$ , assume we have already found a Baire subset in  $\mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$  for which all branched covers  $u := v \circ \varphi$  with  $v : (\Sigma, j) \rightarrow (M, J)$  a simple curve of index 0 and  $\deg(\varphi) \leq d - 1$  have  $\mathbf{D}_u^N$  injective. Suppose  $\varphi \in \mathcal{M}_{\mathbf{b}}^d(j)$  has  $r \geq 0$  critical values and  $\deg(\varphi) = d$  and  $\mathbf{D}_u^N$  is not injective for  $u := v \circ \varphi$ . Then picking the minimal regular presentation  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$  for  $\varphi$  and decomposing  $\rho$  into irreducible representations  $\theta_1^{\oplus \ell_1} \oplus \dots \oplus \theta_N^{\oplus \ell_N}$  of  $G$  splits  $\mathbf{D}_u^N$  into twisted Cauchy-Riemann operators  $\dot{\mathbf{D}}_{u,\theta_i}^N$  for  $i = 1, \dots, N$  with

$$k_i := \dim \ker \dot{\mathbf{D}}_{u,\theta_i}^N,$$

and at least one of the  $k_i$  must be strictly positive by Lemma 2.7. If  $k_i > 0$  and  $\theta_i$  is non-faithful, then Lemma 2.11 identifies  $\dot{\mathbf{D}}_{u,\theta_i}^N$  with a summand of  $\mathbf{D}_{u_0}^N$  for some other cover  $u_0$  of  $v$  with strictly smaller degree, implying  $\dim \ker \mathbf{D}_{u_0}^N > 0$  and thus violating the inductive hypothesis. We can therefore assume  $k_i > 0$  for some faithful representation  $\theta_i$ . But then Theorem D and Lemma 2.10 imply that  $u$  lives in a submanifold of the  $2r$ -dimensional space of branched covers of  $v$  with branching data  $\mathbf{b}$ , having dimension at most

$$2r - t_i k_i \left[ k_i - \text{ind}(\dot{\mathbf{D}}_{u,\theta_i}^N) \right] \leq 2r - t_i k_i [k_i + (n - 1)r] = r[2 - t_i k_i (n - 1)] - t_i k_i^2 < 0$$

since we are assuming  $n \geq 3$ . This gives a contradiction and thus completes the induction.  $\square$

In dimension four, the above argument fails to exclude the possibility of  $\dim \ker \dot{\mathbf{D}}_{u,\theta_i}^N = 1$  for some real-type representation  $\theta_i$ , and this is why we do not know whether super-rigidity always holds in dimension four. We will prove in §7 that it does hold for covers of genus zero and one curves, using different techniques based on intersection theory.

*Proof of Theorem B.* Suppose  $v : (\Sigma, j) \rightarrow (M, J)$  is a simple curve intersecting  $\mathcal{U}$  and  $\varphi : (\Sigma', j') \rightarrow (\Sigma, j)$  is a  $d$ -fold unbranched cover for which  $u := v \circ \varphi$  is not regular, hence by Prop. 2.3,  $\mathbf{D}_u^N$  is not surjective. Fixing the minimal regular presentation of  $\varphi$  and considering the splitting (2.4), we find a twisted Cauchy-Riemann operator  $\dot{\mathbf{D}}_{u,\theta_i}^N$  with

$$c_i := \dim \text{coker } \dot{\mathbf{D}}_{u,\theta_i}^N > 0$$

for some irreducible representation  $\theta_i$  of the generalized automorphism group  $G$  of  $\varphi$ . Suppose  $v$  has exactly  $m \geq 0$  critical points, with critical orders  $\ell_1, \dots, \ell_m$ , so viewing these as marked points allows us to consider  $v$  as an element in the space  $\mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m)$ , which has dimension

$$\dim \mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m) = \text{ind}(v) + 2m - 2nZ(dv) \geq 0.$$

The count of critical points  $Z(dv)$  also appears in the relation between  $\text{ind}(v)$  and  $\text{ind } \mathbf{D}_v^N$ : indeed, writing  $v^*TM = T_v \oplus N_v$ , we can view  $dv$  as a holomorphic section of  $\text{Hom}_{\mathbb{C}}(T\Sigma, T_v)$ , hence

$$Z(dv) = c_1(\text{Hom}_{\mathbb{C}}(T\Sigma, T_v)) = -c_1(T\Sigma) + c_1(T_v) = -\chi(\Sigma) + c_1(T_v),$$

implying  $c_1(N_u) = c_1(v^*TM) - c_1(T_v) = c_1(v^*TM) - \chi(\Sigma) - Z(dv)$ . Plugging in this into the Riemann-Roch formula then gives

$$\begin{aligned} \text{ind } \mathbf{D}_v^N &= (n-1)\chi(\Sigma) + 2c_1(N_v) = (n-3)\chi(\Sigma) + 2c_1(v^*TM) - 2Z(dv) \\ &= \text{ind}(v) - 2Z(dv). \end{aligned}$$

Meanwhile,  $\varphi$  lives in the discrete stratum of the space of branched covers since it has no branch points, and Lemma 2.10 reduces to an equality

$$\text{ind } \dot{\mathbf{D}}_{u, \theta_i}^N = \dim W \cdot \text{ind}(\mathbf{D}_v^N).$$

Now using Theorem D, we find that if  $J$  is generic,  $u$  lives in a manifold of dimension at most

$$\begin{aligned} &\dim \mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m) - t_i c_i (c_i + \text{ind } \dot{\mathbf{D}}_{u, \theta_i}^N) \\ &= \text{ind}(v) + 2m - 2nZ(dv) - t_i c_i (c_i + \dim W \cdot \text{ind } \mathbf{D}_v^N) \\ &= \text{ind}(v) + 2m - 2nZ(dv) - t_i c_i (c_i + \dim W \cdot [\text{ind}(v) - 2Z(dv)]) \\ &= (1 - t_i c_i \dim W) [\text{ind}(v) + 2m - 2nZ(dv)] \\ &\quad - 2t_i c_i \dim W \cdot [(n-1)Z(dv) - m] - t_i c_i^2 < 0, \end{aligned}$$

where we note that  $(n-1)Z(dv) - m \geq 0$  since  $n \geq 2$  and every critical point has order at least 1.  $\square$

*Proof of Theorem C.* Assume  $v : (\Sigma, j) \rightarrow (M, J)$  is simple and satisfies  $\text{ind}(v) \geq (n-1)r$ , while  $\varphi : (\Sigma', j') \rightarrow (\Sigma, j)$  has degree  $d \in \mathbb{N}$  and  $r$  critical values. If  $J$  is generic, then by Proposition A.1 the moduli space containing  $v$  has an open and dense subset consisting of immersed curves, so we are free to assume  $v$  is immersed and thus  $\text{ind}(v) = \text{ind } \mathbf{D}_v^N$ . The key observation is then that by Lemma 2.10, the twisted operators  $\dot{\mathbf{D}}_{u, \theta}^N$  all have nonnegative index, hence Theorem D implies that all of them are surjective unless  $(v, \varphi)$  lies in a countable union of submanifolds with positive codimension.  $\square$

**2.4. Some remarks on wall crossing.** Part of the point of Taubes's twisted bundle setup in [Tau96a] was to understand bifurcations of isolated  $J$ -holomorphic tori under generic 1-parameter deformations in  $J$ . While bifurcation theory is not the main topic of this article, it should be clear that such a theory could be developed based on Theorem D, thus we take this opportunity to make a few observations about it.

If  $\{J_s\}_{s \in [0,1]}$  is a generic homotopy of compatible almost complex structures whose endpoints are generic, then as mentioned in Remark 2.9, one can modify Theorem D to the statement that the parametric moduli space

$$\mathcal{M}_{\mathbf{b},G}^d(\mathcal{M}_{g,m}(A, \{J_s\}; \ell_1, \dots, \ell_m))$$

consisting of pairs  $(u, s)$  where  $s \in [0, 1]$  and  $u \in \mathcal{M}_{\mathbf{b},G}^d(\mathcal{M}_{g,m}(A, J_s; \ell_1, \dots, \ell_m))$  contains certain smooth submanifolds characterized by the dimensions of the kernels and cokernels of twisted Cauchy-Riemann operators, and their codimensions are given by the same formula. In this setting, suppose  $\{v_\tau\}$  is a smooth 1-parameter family of simple  $J_{s(\tau)}$ -holomorphic curves with index 0 for some function  $s(\tau) \in [0, 1]$ , and  $\{u_\tau = v_\tau \circ \varphi_\tau\}$  defines a corresponding 1-parameter family of unbranched covers. The latter have index 0 and will be regular for almost every  $\tau$ , but a bifurcation or “wall crossing” phenomenon occurs at any parameter value  $\tau_0$  for which the family  $\{u_\tau\}$  passes (necessarily transversely) through one of the codimension 1 strata given by Theorem D. When this happens, most of the twisted operators  $\dot{\mathbf{D}}_{u_{\tau_0}, \theta}^N$  remain both injective and surjective, but there will be exactly one irreducible representation  $\theta$  for which

$$\dim \ker \dot{\mathbf{D}}_{u_{\tau_0}, \theta}^N = \dim \operatorname{coker} \dot{\mathbf{D}}_{u_{\tau_0}, \theta}^N = 1,$$

and  $\theta$  is necessarily of real type. Whenever  $\theta$  is not faithful, one can factor  $\varphi_\tau$  through a cover  $\hat{\varphi}_\tau$  of smaller degree and instead examine  $\hat{u}_\tau := v_\tau \circ \hat{\varphi}_\tau$ , so that  $\theta$  becomes faithful without loss of generality (cf. Lemma 2.11). For the trivial representation, this means replacing  $u_\tau$  with  $v_\tau$  itself, so regularity fails for the underlying simple curve at  $\tau = \tau_0$ : as shown in [Tau96a], this is the case where the family  $\{v_\tau\}$  undergoes a *birth-death* bifurcation. The other interesting phenomenon examined by Taubes was the *degree-doubling* bifurcation, in which  $v_\tau$  remains regular but it has a double cover  $u_\tau = v_\tau \circ \varphi_\tau$  which loses regularity at  $\tau = \tau_0$ , causing an additional 1-parameter family of simple curves  $\{w_\tau\}$  to collide with  $\{u_\tau\}$  at  $\tau = \tau_0$ . This is what happens when  $\dot{\mathbf{D}}_{u_\tau, \theta}^N$  remains an isomorphism for the trivial representation but acquires 1-dimensional kernel and cokernel for the nontrivial irreducible representation of  $\mathbb{Z}_2$ .

In [Tau96a], no further bifurcations beyond these two types are possible: this can be attributed to the fact that since Taubes only considers unbranched covers of tori, all covers are regular and abelian. As a consequence, all the complex irreducible representations in the picture are 1-dimensional, implying that the only faithful real-type irreducible representations one needs to consider are the trivial representation of the trivial group and the nontrivial representation of  $\mathbb{Z}_2$ . We should not expect this fortunate situation to hold more generally: for unbranched covers with higher genus, one certainly encounters generalized automorphism groups that are non-abelian and thus have faithful real-type representations of dimension greater than one. These should presumably give rise to bifurcation phenomena involving covers of arbitrarily high degree.

In the context of super-rigidity, it is also important to consider bifurcations that involve branched covers of index 0 curves under generic homotopies of  $J$ . Inspecting the proof of Theorem A, one should expect to see interesting phenomena whenever the dimension that was estimated at the end of the proof turns out to be at least  $-1$ , i.e.

$$2r - t_i k_i \left[ k_i - \operatorname{ind}(\dot{\mathbf{D}}_{u, \theta_i}^N) \right] \geq -1.$$

Assuming we're in dimension at least six, this can only mean  $t_i = k_i = 1$  and either  $r = 0$  or  $n = 3$ . The case  $r = 0$  means the cover is unbranched, so this is what we discussed in the previous paragraphs. Bifurcations involving branched covers can evidently also occur in dimension six, and in this case the improved index bound from Lemma 2.10 must be an equality. The scenario is therefore that the rank of the obstruction bundle over the space of covers  $\{v_\tau \circ \varphi_\tau\}$  jumps up by 1 at a particular parameter value  $\tau = \tau_0$  and for some isolated element  $\varphi_{\tau_0}$  in the space of branched covers with only simple (i.e. two-to-one) branch points: this can presumably cause both a change in the Euler number of the obstruction bundle and the breaking off of a new family of simple curves from  $v_{\tau_0} \circ \varphi_{\tau_0}$ . Once again the irreducible representation involved must be of real type but can have arbitrary dimension, meaning we should not expect any limitation on the degree of  $\varphi_{\tau_0}$ , contrary to the situation in [Tau96a].

### 3. SPLITTING CAUCHY-RIEMANN OPERATORS WITH SYMMETRIES

In this section we give a detailed account of the twisted bundle formalism behind Theorem D and prove several lemmas required for its proof, as well as Lemma 2.11. Instead of talking directly about  $J$ -holomorphic curves, we shall work in the context of abstract Cauchy-Riemann operators on vector bundles and their pullbacks.

**3.1. Regular presentations of branched covers.** The following lemma allows us to move freely back and forth between talking about holomorphic branched covers of closed Riemann surfaces and honest covering maps of punctured surfaces.

**Lemma 3.1.** *Suppose  $(\dot{\Sigma}, j)$  is the complement of a finite set of points  $\Theta$  in a closed connected Riemann surface  $(\Sigma, j)$ ,  $(\dot{\Sigma}', j')$  is a connected noncompact Riemann surface, and*

$$\varphi : (\dot{\Sigma}', j') \rightarrow (\dot{\Sigma}, j)$$

*is a holomorphic covering map of finite degree. Then there exists a closed connected Riemann surface  $(\Sigma', j')$  with a finite set of points  $\Theta' \subset \Sigma'$  such that  $(\dot{\Sigma}', j')$  admits a biholomorphic identification with  $(\Sigma' \setminus \Theta', j')$  and  $\varphi$  extends over the punctures to a holomorphic branched cover  $\varphi : (\Sigma', j') \rightarrow (\Sigma, j)$  with  $\varphi^{-1}(\Theta) = \Theta'$ .*

*Proof.* Choose holomorphic coordinate charts identifying neighborhoods  $\mathcal{U}_z \subset (\Sigma, j)$  of each  $z \in \Theta$  with the standard unit disk  $\mathbb{D} \subset \mathbb{C}$  such that  $z$  is mapped to 0, and use the transformation  $(s, t) \mapsto e^{-2\pi(s+it)}$  to define holomorphic cylindrical coordinates  $(s, t) \in [0, \infty) \times S^1$  on  $\dot{\mathcal{U}}_z := \mathcal{U}_z \setminus \{z\} = \mathbb{D} \setminus \{0\}$ . Since the degree of the cover is finite, each lift to  $\dot{\Sigma}'$  of a loop of the form  $t \mapsto (s, t) \in \dot{\mathcal{U}}_z$  for  $s \geq 0$  closes up after some finite number  $k$  of iterations, so that the corresponding connected component of  $\varphi^{-1}(\dot{\mathcal{U}}_z)$  can be identified with  $[0, \infty) \times S^1$  and  $\varphi$  on this component takes the form

$$[0, \infty) \times S^1 \xrightarrow{\varphi} [0, \infty) \times S^1 \subset \dot{\mathcal{U}}_z : (s, t) \mapsto (s, kt).$$

Rescaling the  $s$ -coordinate on  $\varphi^{-1}(\dot{\mathcal{U}}_z)$  to turn this map into  $(s, t) \mapsto (ks, kt)$ ,  $\varphi$  then pulls  $j$  back to the standard complex structure on  $[0, \infty) \times S^1 \subset \varphi^{-1}(\dot{\mathcal{U}}_z)$ , so that  $\dot{\Sigma}'$  is now presented as a compact Riemann surface with cylindrical ends and can thus be identified with a punctured Riemann surface  $(\Sigma' \setminus \Theta', j')$ .  $\square$



Assume  $\varphi : (\Sigma', j') \rightarrow (\Sigma, j)$  is a  $d$ -fold holomorphic branched cover of closed connected Riemann surfaces with branching data  $\mathbf{b}$  as defined in §2.2, having  $r \geq 0$  distinct critical values. Recall from Definition 2.6 the notion of a regular presentation  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$  for  $\varphi$ , where in particular  $\Theta \subset \Sigma$  is a finite set containing the critical values of  $\varphi$ , giving rise to the punctured surfaces

$$\dot{\Sigma} := \Sigma \setminus \Theta, \quad \dot{\Sigma}' := \Sigma' \setminus \Theta',$$

where  $\Theta' := \varphi^{-1}(\Theta)$ .

**Lemma 3.2.** *There exists a natural bijection between the set of isomorphism classes of regular presentations of  $\varphi$  and the set of pairs  $(\Theta, H)$  where  $\Theta \subset \Sigma$  is a finite subset containing the critical values of  $\varphi$  and  $H$  is a finite-index normal subgroup  $H \subset \pi_1(\dot{\Sigma})$  that is contained in  $\varphi_*(\pi_1(\dot{\Sigma}'))$ . This bijection matches any minimal regular presentation to the smallest possible choice of  $\Theta$  and largest possible choice of  $H$ , i.e. the normal core of  $\varphi_*(\pi_1(\dot{\Sigma}'))$ . Moreover, if  $\varphi$  is regular and  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$  is a minimal regular presentation, then there exists a diffeomorphism  $g : \dot{\Sigma}' \rightarrow \dot{\Sigma}''$  such that  $\pi \circ g = \varphi$ .*

*Proof.* Given a finite set  $\Theta \subset \Sigma$  containing the critical values of  $\varphi$ , pick a base point  $w \in \dot{\Sigma}$  and let  $\tilde{\pi} : \mathcal{U} \rightarrow \dot{\Sigma}$  denote the universal cover, with  $\mathcal{U}$  defined as a space of homotopy classes of paths beginning at  $w$ , so that  $\pi_1(\dot{\Sigma}) := \pi_1(\dot{\Sigma}, w)$  acts naturally on  $\mathcal{U}$  as the group of deck transformations for  $\tilde{\pi}$ . Lifting loops based at  $w$  to paths in  $\dot{\Sigma}'$  then defines a homomorphism

$$\tilde{\rho} : \pi_1(\dot{\Sigma}) \rightarrow S(\varphi^{-1}(w)) : \gamma \mapsto \tilde{\rho}_\gamma$$

so that the covering map  $\dot{\Sigma}' \xrightarrow{\varphi} \dot{\Sigma}$  can be identified with

$$\dot{\Sigma}' = (\mathcal{U} \times \varphi^{-1}(w)) / \pi_1(\dot{\Sigma}) \rightarrow \dot{\Sigma} : [(z, \zeta)] \mapsto \tilde{\pi}(z),$$

where  $\pi_1(\dot{\Sigma})$  acts on  $\mathcal{U}$  by deck transformations and on  $\varphi^{-1}(w)$  via  $\tilde{\rho}$ . We claim that

$$\ker \tilde{\rho} \subset \pi_1(\dot{\Sigma})$$

is the normal core of  $\varphi_*(\pi_1(\dot{\Sigma}'))$ . Indeed, selecting a base point  $w' \in \varphi^{-1}(w) \subset \dot{\Sigma}'$  to define  $\pi_1(\dot{\Sigma}') := \pi_1(\dot{\Sigma}', w')$ , we have

$$\varphi_*(\pi_1(\dot{\Sigma}')) = \left\{ \gamma \in \pi_1(\dot{\Sigma}) \mid \tilde{\rho}_\gamma(w') = w' \right\},$$

which obviously contains  $\ker \tilde{\rho}$ . Changing the base point  $w' \in \varphi^{-1}(w)$  changes the subgroup  $\varphi_*(\pi_1(\dot{\Sigma}'))$  by conjugation with arbitrary elements of  $\pi_1(\dot{\Sigma}')$ , and the normal core is the intersection of all these conjugates, which we can now recognize as the intersection of all the stabilizers of the permutation action on  $\varphi^{-1}(w)$ , and that is  $\ker \tilde{\rho}$ .

Suppose  $H \subset \pi_1(\dot{\Sigma})$  is a finite-index normal subgroup contained in  $\varphi_*(\pi_1(\dot{\Sigma}'))$ , and therefore also in  $\ker \tilde{\rho}$ . Then  $\tilde{\rho}$  descends to the finite group  $G := \pi_1(\dot{\Sigma})/H$ , giving a homomorphism

$$\rho : G \rightarrow S(\varphi^{-1}(w)),$$

which is injective if and only if  $H = \ker \tilde{\rho}$ . We can then define a regular presentation  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, \varphi^{-1}(w), f)$  of  $\varphi$  with  $\pi$  as the natural quotient projection

$$\dot{\Sigma}'' := \mathcal{U}/H \xrightarrow{\pi} \mathcal{U}/\pi_1(\dot{\Sigma}) = \dot{\Sigma}$$

and

$$\dot{\Sigma}' = (\mathcal{U} \times \varphi^{-1}(w)) \Big/ \pi_1(\dot{\Sigma}) \xrightarrow{f} (\dot{\Sigma}'' \times \varphi^{-1}(w)) \Big/ G$$

defined via the quotient projection  $\mathcal{U} \rightarrow \mathcal{U}/H = \dot{\Sigma}''$ . Observe that if we choose  $H = \ker \tilde{\rho}$  and  $\varphi$  is regular, then  $\varphi_*(\pi_1(\dot{\Sigma}')) \subset \pi_1(\dot{\Sigma})$  is normal and is therefore identical to  $H$ , so the natural identification of  $\dot{\Sigma}'$  with  $\mathcal{U}/\varphi_*(\pi_1(\dot{\Sigma}')) = \mathcal{U}/H = \dot{\Sigma}''$  gives an isomorphism between the covering maps  $\varphi$  and  $\pi$ .

Finally, suppose  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$  is a regular presentation of  $\varphi$ , and define the subgroup  $H := \pi_*(\pi_1(\dot{\Sigma}''))$ , which is normal since  $\pi : \dot{\Sigma}'' \rightarrow \dot{\Sigma}$  is regular and has finite index since  $\text{Aut}(\pi) = G = \pi_1(\dot{\Sigma})/H$  is finite. We claim  $H \subset \varphi_*(\pi_1(\dot{\Sigma}'))$ : indeed, any  $\gamma \in H$  is represented by a loop  $\dot{\Sigma}$  based at  $w$  that lifts to a loop  $\gamma''$  in  $\dot{\Sigma}''$  and thus has  $d$  lifts to  $\dot{\Sigma}' \cong (\Sigma'' \times I)/G$  in the form  $\gamma \times \{i\}$  for  $i \in I$ . We can therefore use  $H$  to define the regular presentation from the previous paragraph, with  $G = \pi_1(\dot{\Sigma}'')/H$  acting on  $\varphi^{-1}(w)$  via  $\tilde{\rho}$ , and we claim that this is isomorphic to  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ . Indeed, choosing a base point  $w'' \in \pi^{-1}(w) \subset \dot{\Sigma}''$ , the identification  $f : \dot{\Sigma}' \rightarrow (\dot{\Sigma}'' \times I)/G$  provides a bijection

$$\beta : \varphi^{-1}(w) \rightarrow I \quad \text{such that} \quad f(w') = [(w'', \beta(w'))] \text{ for } w' \in \varphi^{-1}(w),$$

and combining this with the natural identification of  $\dot{\Sigma}''$  with  $\mathcal{U}/H$  gives an isomorphism of regular presentations.  $\square$

**Lemma 3.3.** *Suppose  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$  is a minimal regular presentation of  $\varphi : (\Sigma', j') \rightarrow (\Sigma, j)$ , and let  $\pi : (\Sigma'', j'') \rightarrow (\Sigma, j)$  denote the branched cover of closed Riemann surfaces provided by Lemma 3.1 such that  $\dot{\Sigma}'' = \Sigma'' \setminus \pi^{-1}(\Theta)$ . Then for each  $w \in \Theta$  and  $\zeta \in \pi^{-1}(w) \subset \Sigma''$ , the branching order of  $\pi$  at  $\zeta$  is the least common multiple of the branching orders of  $\varphi$  at all  $z \in \varphi^{-1}(w)$ . In particular,  $\pi$  and  $\varphi$  have the same sets of critical values.*

*Proof.* If  $k \in \mathbb{N}$  is the branching order of  $\pi$  at  $\zeta$ , we can find punctured neighborhoods  $\mathcal{U}_w \subset \dot{\Sigma}$  of  $w$  and  $\mathcal{U}_\zeta \subset \dot{\Sigma}''$  of  $\zeta$  and identify both with the half-cylinder  $[0, \infty) \times S^1$  with coordinates  $(s, t)$  such that  $\pi(s, t) = (ks, kt)$ . Let  $G_\zeta \subset G$  denote the group of automorphisms of  $\pi$  that fix  $\zeta$ ; this is necessarily a cyclic group of order  $k$ , with a generator  $g \in G_\zeta$  that acts on  $\mathcal{U}_\zeta \cong [0, \infty) \times S^1$  as the rotation  $(s, t) \mapsto (s, t + 1/k)$ . Then since  $\pi$  is regular, we can restrict the identification  $\dot{\Sigma}' = (\dot{\Sigma}'' \times I)/G$  to  $\mathcal{U}_\zeta$  and obtain an identification

$$\varphi^{-1}(\mathcal{U}_w) = (\mathcal{U}_\zeta \times I) \Big/ G_\zeta.$$

The connected components of  $\varphi^{-1}(\mathcal{U}_w)$  are then in bijective correspondence to the orbits of the  $G_\zeta$ -action on  $I$  defined by  $\rho : G \rightarrow S(I)$ , with the branching order  $k_z \in \mathbb{N}$  of each corresponding point  $z \in \varphi^{-1}(w)$  given by the number of points in its respective orbit in  $I$ . By the orbit-stabilizer theorem, all of these numbers  $k_z$  must divide  $k = |G_\zeta|$ . If  $\ell$  is their least common multiple, we conclude that  $g^\ell \in G_\zeta$  acts trivially on  $I$ , which means  $g^\ell$  is the identity since  $\rho : G \rightarrow S(I)$  is injective for the minimal regular presentation, hence  $\ell = k$ .  $\square$

It will be important to understand how the various objects constructed out of a regular presentation vary smoothly under changes in  $\varphi$  and  $j$ . To this end, we shall fix the following data for the remainder of §3:

- $\varphi : (\Sigma', j') \rightarrow (\Sigma, j)$  is a holomorphic branched cover of degree  $d \in \mathbb{N}$  with branching data  $\mathbf{b}$ ;
- $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$  is a regular presentation of  $\varphi$ ;
- $P$  is a connected smooth Banach manifold;
- $\mathcal{V} \subset \dot{\Sigma}$  is an open subset with compact closure;
- $\{j_\tau\}_{\tau \in P}$  is a smooth family of complex structures on  $\Sigma$  that match  $j$  outside of  $\mathcal{V}$ ;
- $\{\psi_\tau\}_{\tau \in P}$  is a smooth family of diffeomorphisms  $\psi_\tau : \Sigma \rightarrow \Sigma$  which restrict to the identity on  $\mathcal{V}$  and are  $j$ -holomorphic near  $\Theta$ .

We shall abbreviate the family of closed Riemann surfaces determined by  $j_\tau$  as

$$\Sigma_\tau := (\Sigma, j_\tau),$$

and denote by

$$\pi : (\Sigma'', j'') \rightarrow (\Sigma, j), \quad \Theta'' = \pi^{-1}(\Theta) \subset \Sigma''$$

the holomorphic branched cover of closed Riemann surfaces provided by Lemma 3.1 such that  $\dot{\Sigma}'' = \Sigma'' \setminus \Theta''$ . These choices produce a family of punctured Riemann surfaces

$$\dot{\Sigma}_\tau := (\Sigma \setminus \Theta_\tau, j_\tau) \quad \text{where} \quad \Theta_\tau := \psi_\tau(\Theta) \subset \Sigma,$$

and we define

$$\varphi_\tau := \psi_\tau \circ \varphi : \Sigma' \rightarrow \Sigma, \quad j'_\tau := \varphi_\tau^* j_\tau \text{ on } \Sigma',$$

where we observe that  $j'_\tau$  is always well defined and matches  $j'$  near  $\Theta'$  since  $\psi_\tau$  is holomorphic near  $\Theta$ . This makes

$$\varphi_\tau : \Sigma'_\tau \rightarrow \Sigma_\tau$$

a smooth family of holomorphic branched covers, where

$$\Sigma'_\tau := (\Sigma', j'_\tau),$$

and they restrict to holomorphic covering maps of punctured surfaces  $\dot{\Sigma}'_\tau \xrightarrow{\varphi} \dot{\Sigma}_\tau$ , where

$$\dot{\Sigma}'_\tau := (\dot{\Sigma}', j'_\tau).$$

**Example 3.4.** Suppose  $\Theta$  is the set of critical values of  $\varphi$ ,  $r := |\Theta|$ ,  $P$  is the  $2r$ -dimensional open ball  $B^{2r}$ ,  $j_\tau := j$  for all  $\tau$ , and  $\psi_\tau : \Sigma \rightarrow \Sigma$  is chosen to be any smooth family of diffeomorphisms supported near  $\Theta$  that are holomorphic in a smaller neighborhood of  $\Theta$  and such that  $\psi_0 = \text{Id}$  and

$$B^{2r} \rightarrow \Sigma^{\times r} : \tau \mapsto (\psi_\tau(w_1), \dots, \psi_\tau(w_r))$$

is an embedding onto an open subset, where  $\Theta = \{w_1, \dots, w_r\}$ . Then the branched covers  $\varphi_\tau : (\Sigma', j'_\tau) \rightarrow (\Sigma, j)$  parametrize a neighborhood of  $\varphi$  in  $\mathcal{M}_{\mathbf{b}}^d(j)$ .

**Example 3.5.** If  $v_0 : (\Sigma, j_0) \rightarrow (M, J_0)$  represents a simple element of the moduli space  $\mathcal{M}_{g,m}(A, J_0; \ell_1, \dots, \ell_m)$  defined in Appendix A and  $J_0$  is generic, then one can enhance the previous example as follows to parametrize a neighborhood of  $u_0 := v_0 \circ \varphi$  in the space  $\mathcal{M}_{\mathbf{b},G}^d(\mathcal{M}_{g,m}(A, J_0; \ell_1, \dots, \ell_m))$ . A neighborhood of  $v_0$  in  $\mathcal{M}_{g,m}(A, J_0; \ell_1, \dots, \ell_m)$  can be identified with a smooth submanifold  $X$  of  $\bar{\partial}_{J_0}^{-1}(0)$ , where  $\bar{\partial}_{J_0} : \mathcal{T} \times \mathcal{B} \rightarrow \mathcal{E}$  is the nonlinear Cauchy-Riemann operator defined on the product of  $\mathcal{B} := W^{k,p}(\Sigma, M)$  with a Teichmüller slice  $\mathcal{T}$  through  $j_0$ , cf. Appendix A. Here  $\mathcal{T}$  is a finite-dimensional smooth

family of complex structures on  $\Sigma$ , which can all be arranged to match  $j_0$  near  $\Theta$ . A neighborhood in  $\mathcal{M}_{\mathbf{b},G}^d(\mathcal{M}_{g,m}(A, J_0; \ell_1, \dots, \ell_m))$  is now parametrized by

$$P := B^{2r} \times X,$$

namely via the curves  $v \circ (\psi_\sigma \circ \varphi) : (\Sigma', \varphi^* \psi_\sigma^* j) \rightarrow (M, J_0)$  for each  $\tau := (\sigma, (j, v)) \in P$ , and we associate to these parameters the families  $j_\tau := j$  and  $\psi_\tau := \psi_\sigma$ .

**Example 3.6.** Enhancing the previous example one step further, suppose  $\mathcal{J}_\varepsilon$  is an infinite-dimensional Banach manifold consisting of smooth almost complex structures and we consider a neighborhood of  $(v_0, J_0)$  in the universal moduli space

$$\mathcal{U}^*(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m) = \{(v, J) \mid J \in \mathcal{J}_\varepsilon, v \in \mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m)\}.$$

Such a neighborhood can be identified with a finite-codimensional submanifold  $X$  in the infinite-dimensional Banach manifold  $\bar{\partial}^{-1}(0) \subset \mathcal{T} \times \mathcal{B} \times \mathcal{J}_\varepsilon$ , where  $\bar{\partial}(j, u, J) := \bar{\partial}_J(j, u)$ . Defining  $P := B^{2r} \times X$  and the families  $\{j_\tau\}$  and  $\{\psi_\tau\}$  as in Example 3.5, the parameter space  $P$  is now infinite dimensional.

Observe that the branched covers in the family  $\varphi_\tau$  all have essentially the same topological properties, e.g. their branch points and automorphism groups are identical. It is therefore trivial to extend  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$  to a smooth family of regular presentations

$$(\Theta_\tau, \dot{\Sigma}'', \pi_\tau, G, \rho, I, f)$$

for  $\varphi_\tau$ , where  $\pi_\tau := \psi_\tau \circ \pi$ . By the same reasoning as above, we can define on  $\Sigma''$  a smooth family of complex structures  $j_\tau'' := \pi_\tau^* j_\tau$  such that

$$\pi_\tau : \Sigma_\tau'' \rightarrow \Sigma_\tau, \quad \Sigma_\tau'' := (\Sigma'', j_\tau'')$$

becomes a smooth family of holomorphic branched covers, restricting to a smooth family of holomorphic covering maps  $\dot{\Sigma}_\tau'' \xrightarrow{\pi_\tau} \dot{\Sigma}_\tau$ , defined on the family of punctured Riemann surfaces

$$\dot{\Sigma}_\tau'' := (\dot{\Sigma}'', j_\tau'').$$

**Example 3.7.** If  $\varphi$  is regular with  $\text{Aut}(\varphi) = G$ , then it admits a canonical minimal regular presentation  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$  where  $\dot{\Sigma}'' := \dot{\Sigma}'$ ,  $\pi := \varphi$ ,  $I := G$ , and the action  $\rho : G \rightarrow S(G)$  of  $G$  on itself is defined by left multiplication

$$\rho_g(h) := gh.$$

Here the identification  $\dot{\Sigma}' \xrightarrow{f} (\dot{\Sigma}'' \times G)/G$  sends  $z \in \dot{\Sigma}'$  to  $[(z, e)]$ , where  $e \in G$  is the identity element. The action of  $G$  on  $\dot{\Sigma}' = (\dot{\Sigma}'' \times G)/G$  by deck transformations can now be presented as the action via right multiplication

$$G \times \dot{\Sigma}' \rightarrow \dot{\Sigma}' : (g, [(z, h)]) \mapsto [(z, hg^{-1})].$$

Notice that any regular presentation in which  $\rho : G \rightarrow S(I)$  acts on  $I$  both transitively and without fixed points is isomorphic to one of this form, since for any  $i \in I$ , the map  $G \rightarrow I : g \mapsto \rho_g(i)$  defines a bijection that transforms the action by left multiplication into  $\rho$ .

**Example 3.8.** The following construction underlies Lemma 2.11: any proper normal subgroup  $H \subset G$  gives rise to a factorization of  $\varphi : (\Sigma', j') \rightarrow (\Sigma, j)$  in the following way. Let  $I/H$  denote the set of orbits for the action  $\rho|_H : H \rightarrow S(I)$ . Then  $G/H$  is a finite group and  $\rho$  descends to a homomorphism

$$\rho_H : G/H \rightarrow S(I/H),$$

which acts transitively on  $I/H$ . The regular cover  $\pi : \dot{\Sigma}'' \rightarrow \dot{\Sigma} = \dot{\Sigma}''/G$  now factors through the obvious projections

$$\dot{\Sigma}'' \rightarrow \dot{\Sigma}_H'' := \dot{\Sigma}''/H \xrightarrow{\pi_H} \dot{\Sigma} = \dot{\Sigma}''/G,$$

and  $\pi_H : \dot{\Sigma}_H'' \rightarrow \dot{\Sigma}$  is a regular holomorphic cover with automorphism group  $G/H$ . We can thus define

$$\dot{\Sigma}_H' := \left( \dot{\Sigma}_H'' \times (I/H) \right) / (G/H) \xrightarrow{\varphi_H} \dot{\Sigma} : [(z, i)] \mapsto \pi_H(z),$$

as well as a factorization of  $\varphi : \dot{\Sigma}' \rightarrow \dot{\Sigma}$  by covering maps

$$\dot{\Sigma}' = \left( \dot{\Sigma}'' \times I \right) / G \longrightarrow \dot{\Sigma}_H' \xrightarrow{\varphi_H} \dot{\Sigma},$$

where the first map is also defined via the obvious quotient projections. Lemma 3.1 implies that  $\dot{\Sigma}_H'$  and  $\dot{\Sigma}_H''$  each arise by puncturing closed connected Riemann surfaces  $(\Sigma_H', j_H')$  and  $(\Sigma_H'', j_H'')$  respectively, and in particular we obtain a factorization of  $\varphi$  via holomorphic branched covers

$$(\Sigma', j') \rightarrow (\Sigma_H', j_H') \xrightarrow{\varphi_H} (\Sigma, j)$$

with  $\deg(\varphi_H) \leq d$  equal to the number of distinct orbits of the  $H$ -action on  $I$ , hence

$$\deg(\varphi_H) < d$$

holds whenever the action of  $H$  on  $I$  is nontrivial. Note that  $\varphi_H$  inherits from this construction a regular presentation  $(\Theta, \dot{\Sigma}_H'', \pi_H, G/H, \rho_H, I/H, f_H)$ , though it need not be minimal and  $\Theta$  may contain points that are not critical values of  $\varphi_H$ , even if  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$  is minimal. This is the main reason why non-minimal regular presentations have been included in the discussion.

**3.2. Cauchy-Riemann operators on closed and punctured domains.** Fix a complex vector bundle

$$(E, J) \rightarrow (\Sigma, j)$$

of rank  $m \geq 1$ , and define the rank  $m$  bundle of complex-antilinear maps

$$F = \overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, E) = \Lambda^{0,1} T^* \Sigma \otimes E.$$

Recall that a first-order real-linear partial differential operator  $\mathbf{D} : \Gamma(E) \rightarrow \Gamma(F) = \Omega^{0,1}(\Sigma, E)$  is then called a **Cauchy-Riemann type operator** on  $E$  if it satisfies the Leibniz rule

$$\mathbf{D}(f\eta) = (\bar{\partial}f)\eta + f\mathbf{D}\eta$$

for all  $\eta \in \Gamma(E)$  and  $f \in C^\infty(\Sigma, \mathbb{R})$ , where  $\bar{\partial}f = df + i df \circ j \in \Omega^{0,1}(\Sigma)$ . The space

$$\mathcal{CR}_{\mathbb{R}}(E)$$

of all such operators is an affine space modelled on the space of smooth real-linear bundle maps  $\Gamma(\text{Hom}_{\mathbb{R}}(E, F)) = \Omega^{0,1}(\Sigma, \text{End}_{\mathbb{R}}(E, J))$ . The **pullback** of  $\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}(E)$  via  $\varphi : (\Sigma', j') \rightarrow (\Sigma, j)$  defines a Cauchy-Riemann operator

$$\varphi^* \mathbf{D} : \Gamma(E^\varphi) \rightarrow \Gamma(F^\varphi),$$

where we define two bundles over  $\Sigma'$  by

$$E^\varphi := \varphi^* E, \quad F^\varphi := \overline{\text{Hom}_{\mathbb{C}}}(T\Sigma', \varphi^* E) = \Lambda^{0,1} T^* \Sigma' \otimes E^\varphi$$

and characterize  $\varphi^* \mathbf{D}$  via the relation

$$(\varphi^* \mathbf{D})(\eta \circ \varphi) = \varphi^* (\mathbf{D}\eta) \quad \text{for all } \eta \in \Gamma(E).$$

**Example 3.9.** If  $v : (\Sigma, j) \rightarrow (M, J)$  is a  $J$ -holomorphic curve with generalized normal bundle  $N_v \rightarrow \Sigma$ , its normal Cauchy-Riemann operator  $\mathbf{D}_v^N$  belongs to  $\mathcal{CR}_{\mathbb{R}}(N_v)$ , and if  $u = v \circ \varphi : (\Sigma', j') \rightarrow (M, J)$ , then  $N_u = \varphi^* N_v$  and  $\mathbf{D}_u^N = \varphi^* \mathbf{D}_v^N \in \mathcal{CR}_{\mathbb{R}}(N_u)$ .

Fixing Hermitian bundle metrics  $\langle \cdot, \cdot \rangle_E$  and  $\langle \cdot, \cdot \rangle_\Sigma$  on  $E$  and  $T\Sigma$  respectively, we can integrate real parts of bundle metrics to define real-valued  $L^2$ -pairings  $\langle \cdot, \cdot \rangle_{L^2}$  on  $\Gamma(E)$  and  $\Gamma(F)$ , which determines a **formal adjoint** operator  $\mathbf{D}^* : \Gamma(F) \rightarrow \Gamma(E)$  via the relation

$$\langle \alpha, \mathbf{D}\eta \rangle_{L^2} = \langle \mathbf{D}^* \alpha, \eta \rangle_{L^2}$$

for all smooth sections  $\alpha \in \Gamma(F)$  and  $\eta \in \Gamma(E)$  with compact support.<sup>1</sup> Viewing  $\mathbf{D}$  as a Fredholm operator on Sobolev spaces  $W^{k,p}(E) \rightarrow W^{k-1,p}(F)$  for some  $k \in \mathbb{N}$  and  $p \in (1, \infty)$ , we can then identify  $\text{coker } \mathbf{D}$  with  $\ker \mathbf{D}^* \subset \Gamma(F)$ , which is the  $L^2$ -orthogonal complement of  $\text{im } \mathbf{D} \subset W^{k-1,p}(F)$  and is a finite-dimensional space of smooth sections by elliptic regularity. Using the Riemann-Roch formula  $\text{ind}(\mathbf{D}) = m\chi(\Sigma) + 2c_1(E)$  and computing the algebraic count of branch points  $Z(d\varphi)$  from the Riemann-Hurwitz formula, the (real) Fredholm indices of  $\mathbf{D}$  and  $\varphi^* \mathbf{D}$  are related by

$$\text{ind}(\varphi^* \mathbf{D}) = d \cdot \text{ind } \mathbf{D} - mZ(d\varphi).$$

For technical reasons that will emerge in §5, we will sometimes want to explicitly exclude the possibility of  $\mathbf{D}$  being complex linear. Note that any  $\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}(E)$  has a complex-linear part which is also a Cauchy-Riemann operator

$$\mathbf{D}_{\mathbb{C}} := \frac{1}{2}(\mathbf{D} - J \circ \mathbf{D} \circ J) \in \mathcal{CR}_{\mathbb{R}}(E),$$

hence its complex-antilinear part is a tensorial bundle map

$$\mathbf{D}_{\overline{\mathbb{C}}} := \frac{1}{2}(\mathbf{D} + J \circ \mathbf{D} \circ J) \in \Gamma(\overline{\text{Hom}_{\mathbb{C}}}(E, F)).$$

**Definition 3.10.** Denote by

$$\mathcal{CR}_{\mathbb{R}}^*(E) \subset \mathcal{CR}_{\mathbb{R}}(E)$$

the subset consisting of operators  $\mathbf{D}$  such that the complex-antilinear part  $\mathbf{D}_{\overline{\mathbb{C}}} : E \rightarrow F$  is invertible on some fiber.

---

<sup>1</sup>The compact support condition is vacuous in the present context since  $\Sigma$  is compact, but the same definition is also valid on punctured domains.



The space  $\mathcal{CR}_{\mathbb{R}}^*(E)$  is open and dense in  $\mathcal{CR}_{\mathbb{R}}(E)$  in the  $C_{\text{loc}}^\infty$ -topology, as one can always perturb the antilinear part to make it invertible on some open set. Moreover, whenever  $\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}^*(E)$ , we automatically have  $\varphi^*\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}^*(E^\varphi)$ . Recall that a real subspace  $W$  of a complex vector space  $(V, J)$  is called **totally real** if  $W \cap JW = \{0\}$ .<sup>2</sup>

**Lemma 3.11.** *If  $\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}^*(E)$ , then  $\ker \mathbf{D}$  and  $\ker \mathbf{D}^*$  are totally real subspaces of  $\Gamma(E)$  and  $\Gamma(F)$  respectively.*

*Proof.* If  $\eta \in \ker \mathbf{D}$  and  $J\eta \in \ker \mathbf{D}$ , then  $\mathbf{D}_\mathbb{C}\eta \equiv 0$  and thus  $\mathbf{D}_{\overline{\mathbb{C}}}\eta \equiv 0$ , but the latter is impossible if  $\eta$  is nontrivial and  $\mathbf{D}_{\overline{\mathbb{C}}} : E \rightarrow F$  is invertible on any open subset, since unique continuation implies that  $\eta$  cannot vanish identically on that subset. The corresponding statement about  $\ker \mathbf{D}^*$  follows since the antilinear part of  $\mathbf{D}^*$  is the transpose of the antilinear part of  $\mathbf{D}$  and is thus invertible on any fiber where the latter is invertible.  $\square$

In order to exploit the topological constructions in the previous section, we will need to work with Cauchy-Riemann type operators on punctured surfaces instead of closed surfaces. We shall now show that this can be done without loss of generality by choosing suitable weighted Sobolev spaces. Assume

$$E_\tau \rightarrow \Sigma_\tau$$

is a smooth family of rank  $m$  complex vector bundles with complex structures  $J_\tau$ , equipped with a smooth family of Cauchy-Riemann operators  $\mathbf{D}_\tau \in \mathcal{CR}_{\mathbb{R}}(E_\tau)$ . Denote the restrictions of the bundles  $E_\tau$  and

$$F_\tau := \overline{\text{Hom}}_{\mathbb{C}}(T\Sigma_\tau, E_\tau)$$

to the punctured surfaces  $\dot{\Sigma}_\tau$  by

$$\dot{E}_\tau := E_\tau|_{\dot{\Sigma}_\tau}, \quad \dot{F}_\tau := F_\tau|_{\dot{\Sigma}_\tau} = \overline{\text{Hom}}_{\mathbb{C}}(T\dot{\Sigma}_\tau, \dot{E}_\tau).$$

Restricting  $\mathbf{D}_\tau$  to  $\dot{\Sigma}_\tau$  then defines a family of Cauchy-Riemann type operators

$$\dot{\mathbf{D}}_\tau \in \mathcal{CR}_{\mathbb{R}}(\dot{E}_\tau).$$

In order to understand the functional-analytic properties of  $\dot{\mathbf{D}}_\tau$ , we must examine its asymptotic behavior fairly carefully. Fix local holomorphic coordinate charts to identify a neighborhood of each  $w \in \Theta$  in  $\Sigma$  with the closed unit disk  $\mathbb{D} \subset \mathbb{C}$ , with  $w$  corresponding to  $0 \in \mathbb{D}$ , and use  $\psi_\tau$  to produce from these a smooth family of holomorphic charts on neighborhoods of  $\psi_\tau(w) \in \Theta_\tau$  for  $\tau \in P$ . In these coordinates, use the biholomorphic map

$$[0, \infty) \times S^1 \rightarrow \mathbb{D} \setminus \{0\} : (s, t) \mapsto e^{-2\pi(s+it)}$$

to define cylindrical ends of  $\dot{\Sigma}_\tau$  with holomorphic coordinates  $(s, t) \in [0, \infty) \times S^1$ . Choose also a smooth family of trivializations of  $E_\tau$  near  $\Theta_\tau$  and denote the resulting trivialization of  $\dot{E}_\tau$  over the cylindrical ends by  $\Phi$ . The relative first Chern number<sup>3</sup> of  $\dot{E}_\tau$  is then given

<sup>2</sup>The term “totally real” is sometimes used in the symplectic literature to mean  $V = W \oplus JW$ , but we are not assuming this, e.g. we can allow  $\dim V = \infty$  while  $\dim W < \infty$ .

<sup>3</sup>Recall that for any complex line bundle  $E$  over a surface  $\Sigma$  with a trivialization  $\Phi$  specified outside of some open subset in  $\Sigma$  with compact closure, the relative first Chern number  $c_1^\Phi(E) \in \mathbb{Z}$  is defined by algebraically counting the zeroes of a generic section that is constant with respect to  $\Phi$  wherever the latter is defined. This definition extends uniquely to higher rank bundles via the relation  $c_1^{\Phi_1 \oplus \Phi_2}(E_1 \oplus E_2) = c_1^{\Phi_1}(E_1) + c_1^{\Phi_2}(E_2)$ .

by

$$(3.1) \quad c_1^\Phi(\dot{E}_\tau) = c_1(E_\tau) \in \mathbb{Z}.$$

For any tuple of real numbers

$$\boldsymbol{\delta} = \{\delta_w \in \mathbb{R}\}_{w \in \Theta},$$

we can use the chosen coordinates and trivializations over the cylindrical ends of  $\dot{\Sigma}_\tau$  to define the Sobolev space with **exponential weights**

$$W^{k,p,\boldsymbol{\delta}}(\dot{E}_\tau) := \left\{ \eta \in W_{\text{loc}}^{k,p}(\dot{E}_\tau) \mid e^{\delta_w s} \eta \in W^{k,p}([0, \infty) \times S^1) \text{ on the end near } \psi_\tau(w) \in \Theta_\tau \right\}.$$

We will also write

$$L^{p,\boldsymbol{\delta}}(\dot{E}_\tau) := W^{0,p,\boldsymbol{\delta}}(\dot{E}_\tau).$$

Note that sections  $\eta \in W^{k,p,\boldsymbol{\delta}}(\dot{E}_\tau)$  have exponential decay at any end where  $\delta_w > 0$ , but one can also take  $\delta_w < 0$ , in which case  $\eta$  may be unbounded with exponential *growth* near  $w$ . In order to emphasize when we are using negative exponential weights, we associate to  $\boldsymbol{\delta} = \{\delta_w\}_{w \in \Theta}$  the inverse set of weights

$$-\boldsymbol{\delta} := \{-\delta_w\}_{w \in \Theta}.$$

The asymptotic coordinates and trivializations also naturally give rise to asymptotic trivializations of  $\dot{F}_\tau = \overline{\text{Hom}}_{\mathbb{C}}(T\dot{\Sigma}_\tau, \dot{E}_\tau)$ , so we can similarly define the Banach space  $W^{k-1,p,\boldsymbol{\delta}}(\dot{F}_\tau)$ , which is a completion of some subset of  $\Omega^{0,1}(\dot{\Sigma}_\tau, \dot{E}_\tau)$  determined by the asymptotic conditions.

Choose a smooth  $\tau$ -parametrized family of Hermitian bundle metrics and connections on  $E_\tau$  which match the trivial metric and connection in our chosen family of trivializations near  $\Theta_\tau$ . Any Cauchy-Riemann type operator on  $E_\tau$  can then be written as  $\mathbf{D}_\tau = \bar{\partial}_\nabla + A$  for some  $A \in \Omega^{0,1}(\Sigma, \text{End}_{\mathbb{R}}(E_\tau))$ , where  $\bar{\partial}_\nabla := \nabla + J_\tau \circ \nabla \circ j_\tau : \Gamma(E_\tau) \rightarrow \Omega^{0,1}(\Sigma, E_\tau)$ . In the chosen coordinates and trivialization near a point  $w \in \Theta_\tau$ , the  $(0,1)$ -form  $A$  can be written as

$$A = A_\tau^{(w)}(z) d\bar{z}$$

for some smooth function  $A_\tau^{(w)} : \mathbb{D} \rightarrow \text{End}_{\mathbb{R}}(\mathbb{C}^m)$ . The restriction of  $A$  to an  $\text{End}_{\mathbb{R}}(\dot{E}_\tau)$ -valued  $(0,1)$ -form  $\dot{A}_\tau \in \Omega^{0,1}(\dot{\Sigma}_\tau, \text{End}_{\mathbb{R}}(\dot{E}_\tau))$  can then be written on the corresponding cylindrical end as

$$\dot{A}_\tau = \dot{A}_\tau^{(w)}(s, t) (-ds + i dt)$$

where

$$(3.2) \quad \dot{A}_\tau^{(w)}(s, t) := 2\pi e^{-2\pi(s-it)} A_\tau^{(w)}(e^{-2\pi(s+it)}),$$

and given a section  $\eta \in \Gamma(\dot{E}_\tau)$  expressed as a function  $\eta(s, t) \in \mathbb{C}^m$  with respect to the trivialization on the same end,  $\dot{\mathbf{D}}_\tau \eta$  on this end takes the form

$$(3.3) \quad \dot{\mathbf{D}}_\tau \eta = \left( \bar{\partial} \eta + \dot{A}_\tau^{(w)} \eta \right) (-ds + i dt).$$

Observe that  $\dot{A}_\tau^{(w)}(s, \cdot) \rightarrow 0$  with all derivatives as  $s \rightarrow \infty$ . This expression shows that  $\dot{\mathbf{D}}_\tau$  extends to a bounded linear operator

$$\dot{\mathbf{D}}_\tau : W^{k,p,\boldsymbol{\delta}}(\dot{E}_\tau) \rightarrow W^{k-1,p,\boldsymbol{\delta}}(\dot{F}_\tau)$$

for any choices of  $k \in \mathbb{N}$ ,  $p \in (1, \infty)$  and exponential weights  $\delta = \{\delta_w \in \mathbb{R}\}_{w \in \Theta}$ . Operators of this type are standard in Floer-type theories and especially in symplectic field theory. Appealing to the Fredholm theory on punctured surfaces developed in [Sch95], the asymptotic decay of  $\dot{A}_\tau^{(w)}(s, \cdot)$  means that  $\dot{\mathbf{D}}_\tau : W^{k,p}(\dot{E}_\tau) \rightarrow W^{k-1,p}(\dot{F}_\tau)$  behaves like a linearized Cauchy-Riemann operator for a punctured holomorphic curve with trivial asymptotic operators (corresponding to degenerate periodic orbits), and is thus *not* Fredholm. But it becomes Fredholm when we introduce suitable weights: conjugating  $\dot{\mathbf{D}}_\tau : W^{k,p,\delta} \rightarrow W^{k-1,p,\delta}$  with a map of the form  $\Psi(\eta) = e^f \eta$  for a suitable function  $f : \dot{\Sigma}_\tau \rightarrow \mathbb{R}$  (cf. [HWZ99, §6] or [Wen10, §2.1]) produces a commutative diagram

$$(3.4) \quad \begin{array}{ccc} W^{k,p,\delta}(\dot{E}_\tau) & \xrightarrow{\dot{\mathbf{D}}_\tau} & W^{k-1,p,\delta}(\dot{F}_\tau) \\ \downarrow \Psi & & \downarrow \Psi \\ W^{k,p}(\dot{E}_\tau) & \xrightarrow{\hat{\mathbf{D}}_\tau} & W^{k-1,p}(\dot{F}_\tau), \end{array}$$

where  $\hat{\mathbf{D}} : W^{k,p} \rightarrow W^{k-1,p}$  is another Cauchy-Riemann type operator whose asymptotic operators are offset by constants depending on the weights  $\delta$ , and thus is Fredholm for suitable choices. In particular, the computation in (3.7) and (3.8) below will show that imposing the exponential growth condition  $e^{-\delta s} \eta \in W^{k,p}([0, \infty) \times S^1)$  on each cylindrical end for sufficiently small  $\delta > 0$  adjusts the asymptotic operators of  $\hat{\mathbf{D}}_\tau$  so that each acquires an effective Conley-Zehnder index of  $m$  relative to the trivialization  $\Phi$ .

We need to be a bit cautious with the weights when discussing elliptic regularity and formal adjoints: as a rule, the Sobolev constants  $k \in \mathbb{N}$  and  $p \in (1, \infty)$  can be changed freely, but the weights cannot. The following are immediate consequences of (3.4) after applying standard regularity arguments to  $\hat{\mathbf{D}}_\tau$ , plus the fact Cauchy-Riemann operators with nondegenerate asymptotics automatically impose exponential decay conditions on their kernels (cf. [Sch95, Prop. 3.1.26]):

**Lemma 3.12.** *Suppose  $k \in \mathbb{N}$ ,  $1 < p < \infty$ , and  $\delta = \{\delta_w \in \mathbb{R}\}_{w \in \Theta}$  is any choice of exponential weights. If  $\eta \in L^{p,\delta}(\dot{E}_\tau)$  is a weak solution to  $\dot{\mathbf{D}}_\tau \eta = \xi$  for  $\xi \in W^{k-1,p,\delta}(\dot{F}_\tau)$ , then  $\eta \in W^{k,p,\delta}(\dot{E}_\tau)$ .  $\square$*

**Lemma 3.13.** *Suppose  $1 < p < \infty$  and the weights  $\delta$  are chosen such that  $\dot{\mathbf{D}}_\tau : W^{k,p,\delta}(\dot{E}_\tau) \rightarrow W^{k-1,p,\delta}(\dot{F}_\tau)$  is Fredholm. If  $\eta \in L^{p,\delta}(\dot{E}_\tau)$  is a weak solution to  $\dot{\mathbf{D}}_\tau \eta = 0$ , then  $\eta \in W^{k,q,\delta}(\dot{E}_\tau)$  for all  $k \in \mathbb{N}$  and  $q \in (1, \infty)$ .  $\square$*

To discuss the formal adjoint on punctured domains, one should define real  $L^2$ -products for  $\Gamma(\dot{E}_\tau)$  and  $\Gamma(\dot{F}_\tau)$  in terms of a family of Hermitian bundle metrics on  $E_\tau$  and Riemannian metrics on  $\dot{\Sigma}_\tau$  that are compatible with the conformal structure and standard on the cylindrical ends. The key technical point is then the following: there are well-defined  $L^2$ -pairings

$$(3.5) \quad L^{p,\delta} \otimes L^{q,-\delta} \rightarrow \mathbb{R} : \eta \otimes \xi \mapsto \langle \eta, \xi \rangle_{L^2}$$

whenever  $1/p + 1/q = 1$ , and using the density of  $C_0^\infty$ , the usual relation

$$(3.6) \quad \langle \alpha, \dot{\mathbf{D}}_\tau \eta \rangle_{L^2} = \langle \dot{\mathbf{D}}_\tau^* \alpha, \eta \rangle_{L^2}$$

for smooth sections with compact support remains valid whenever  $\eta \in W^{1,p,-\delta}(\dot{E}_\tau)$  and  $\alpha \in W^{1,q,\delta}(\dot{F}_\tau)$  for  $1/p + 1/q = 1$ . Using (3.4), one finds  $\dot{\mathbf{D}}_\tau^* = \Psi \widehat{\mathbf{D}}_\tau^* \Psi^{-1}$ , from which one can check that  $\dot{\mathbf{D}}_\tau^* : W^{k,p,\delta}(\dot{F}_\tau) \rightarrow W^{k-1,p,\delta}(\dot{E}_\tau)$  satisfies the Fredholm property and Lemmas 3.12 and 3.13 under the same conditions on  $\delta$  as  $\dot{\mathbf{D}}_\tau : W^{k,p,-\delta}(\dot{E}_\tau) \rightarrow W^{k-1,p,-\delta}(\dot{F}_\tau)$ . The next result appears standard at first glance, but the reader should be cautioned that it depends on inclusions  $W^{k,p,\delta} \hookrightarrow W^{k,p,-\delta}$  which hold only when all the weights are nonnegative, so e.g. one does not obtain any similar result with the roles of  $\dot{\mathbf{D}}_\tau$  and  $\dot{\mathbf{D}}_\tau^*$  reversed.

**Proposition 3.14.** *Assume  $k \in \mathbb{N}$ ,  $1 < p < \infty$ , and  $\delta = \{\delta_w \geq 0\}_{w \in \Theta}$  is a set of nonnegative exponential weights such that*

$$\dot{\mathbf{D}}_\tau : W^{k,p,-\delta}(\dot{E}_\tau) \rightarrow W^{k-1,p,-\delta}(\dot{F}_\tau)$$

*is Fredholm. Defining its formal adjoint as a bounded linear map*

$$\dot{\mathbf{D}}_\tau^* : W^{k,p,\delta}(\dot{F}_\tau) \rightarrow W^{k-1,p,\delta}(\dot{E}_\tau)$$

*and using the obvious inclusions  $W^{k,p,\delta}(\dot{F}_\tau) \hookrightarrow W^{k-1,p,\delta}(\dot{F}_\tau) \hookrightarrow W^{k-1,p,-\delta}(\dot{F}_\tau)$ , we have*

$$W^{k-1,p,-\delta}(\dot{F}_\tau) = \text{im } \dot{\mathbf{D}}_\tau \oplus \ker \dot{\mathbf{D}}_\tau^*.$$

*In particular, coker  $\dot{\mathbf{D}}_\tau$  is isomorphic to the space of all sections in  $L^{q,\delta}(\dot{F}_\tau)$  for  $1/p + 1/q = 1$  that are  $L^2$ -orthogonal to  $\text{im } \dot{\mathbf{D}}_\tau \subset L^{p,-\delta}(\dot{F}_\tau)$  under the pairing (3.5).*

*Proof.* If  $\alpha \in \text{im } \dot{\mathbf{D}}_\tau \cap \ker \dot{\mathbf{D}}_\tau^*$ , then  $\alpha = \dot{\mathbf{D}}_\tau \eta$  for some  $\eta \in W^{k,p,-\delta}(\dot{E}_\tau) \subset W^{1,p,-\delta}(\dot{E}_\tau)$ , while  $\alpha$  also belongs to  $W^{1,q,\delta}(\dot{F}_\tau)$  for  $1/p + 1/q = 1$  by Lemma 3.13. Thus  $\alpha$  has a well-defined  $L^2$ -pairing with itself and (3.6) gives

$$\|\alpha\|_{L^2}^2 = \langle \alpha, \dot{\mathbf{D}}_\tau \eta \rangle_{L^2} = \langle \dot{\mathbf{D}}_\tau^* \alpha, \eta \rangle_{L^2} = 0.$$

To show that  $\text{im } \dot{\mathbf{D}}_\tau + \ker \dot{\mathbf{D}}_\tau^*$  is  $W^{k-1,p,-\delta}(\dot{F}_\tau)$ , note first that it is a closed subspace since  $\dot{\mathbf{D}}_\tau$  is Fredholm. Then in the case  $k = 1$ , the contrary would mean there exists a nontrivial  $\lambda \in (L^{p,-\delta}(\dot{F}_\tau))^* = L^{q,\delta}(\dot{F}_\tau)$  for  $1/p + 1/q = 1$  such that  $\langle \dot{\mathbf{D}}_\tau \eta, \lambda \rangle_{L^2} = 0$  for all  $\eta \in W^{1,p,-\delta}(\dot{E}_\tau)$  and  $\langle \alpha, \lambda \rangle_{L^2} = 0$  for all  $\alpha \in \ker \dot{\mathbf{D}}_\tau^*$ . The first condition means  $\lambda \in \ker \dot{\mathbf{D}}_\tau^*$  by Lemma 3.13 and thus contradicts the second unless  $\lambda = 0$ . To extend this result to all  $k \in \mathbb{N}$ , note that if  $\lambda \in W^{k-1,p,-\delta}(\dot{F}_\tau) \subset L^{p,-\delta}(\dot{F}_\tau)$  then the  $k = 1$  case gives  $\eta \in W^{1,p,-\delta}(\dot{E}_\tau)$  and  $\alpha \in \ker \dot{\mathbf{D}}_\tau^*$  such that  $\dot{\mathbf{D}}_\tau \eta + \alpha = \lambda$ . Then Lemma 3.13 implies  $\alpha \in W^{k-1,p,\delta}(\dot{F}_\tau) \subset W^{k-1,p,-\delta}(\dot{F}_\tau)$ , implying that  $\dot{\mathbf{D}}_\tau \eta$  is also in  $W^{k-1,p,-\delta}(\dot{F}_\tau)$ , so Lemma 3.12 implies  $\eta \in W^{k,p,-\delta}(\dot{E}_\tau)$  and we are done.  $\square$

This discussion extends easily to the pulled back operators

$$\varphi_\tau^* \mathbf{D}_\tau \in \mathcal{CR}_\mathbb{R}(\varphi_\tau^* E_\tau) \quad \text{and} \quad \varphi_\tau^* \dot{\mathbf{D}}_\tau \in \mathcal{CR}_\mathbb{R}(\varphi_\tau^* \dot{E}_\tau)$$

on bundles over  $\Sigma'_\tau$  and  $\dot{\Sigma}'_\tau$  respectively. Observe that since  $\dot{\Sigma}'_\tau \xrightarrow{\varphi_\tau} \dot{\Sigma}_\tau$  has no branch points,  $d\varphi_\tau$  gives a bundle isomorphism  $T\dot{\Sigma}'_\tau \rightarrow \varphi_\tau^* T\dot{\Sigma}_\tau$  and we can thus identify

$$F_\tau^{\varphi_\tau}|_{\dot{\Sigma}'_\tau} = \overline{\text{Hom}}_\mathbb{C}(T\dot{\Sigma}'_\tau, \varphi_\tau^* \dot{E}_\tau) = \overline{\text{Hom}}_\mathbb{C}(\varphi_\tau^* T\dot{\Sigma}_\tau, \varphi_\tau^* \dot{E}_\tau) = \varphi_\tau^* \dot{F}_\tau,$$

so that  $\varphi_\tau^* \dot{\mathbf{D}}_\tau$  can be viewed as a map  $\Gamma(\varphi_\tau^* \dot{E}_\tau) \rightarrow \Gamma(\varphi_\tau^* \dot{F}_\tau)$ . We can now define fixed holomorphic cylindrical coordinate systems  $(s, t) \in [0, \infty) \times S^1$  on punctured neighborhoods of each point  $\zeta \in \Theta' = \varphi_\tau^{-1}(\Theta_\tau)$  such that  $\varphi_\tau$  takes the form

$$\dot{\Sigma}'_\tau \supset [0, \infty) \times S^1 \xrightarrow{\varphi_\tau} [0, \infty) \times S^1 \subset \dot{\Sigma}_\tau : (s, t) \mapsto (k_\zeta s, k_\zeta t),$$

where  $k_\zeta \in \mathbb{N}$  is the branching order of  $\varphi$  at  $\zeta$ . Pulling back the trivializations  $\Phi$  on  $E_\tau$  near  $\Theta_\tau$  to define corresponding trivializations of  $\varphi_\tau^* E_\tau$  near  $\Theta'$ , we obtain asymptotic trivializations of  $\varphi_\tau^* \dot{E}_\tau$  and  $\varphi_\tau^* \dot{F}_\tau$  on the cylindrical ends and can thus define weighted Sobolev norms for sections of these bundles, producing a bounded linear operator

$$\varphi_\tau^* \dot{\mathbf{D}}_\tau : W^{k,p,\delta}(\varphi_\tau^* \dot{E}_\tau) \rightarrow W^{k-1,p,\delta}(\varphi_\tau^* \dot{F}_\tau)$$

for all choices of  $k \in \mathbb{N}$ ,  $p \in (1, \infty)$  and exponential weights  $\delta = \{\delta_\zeta \in \mathbb{R}\}_{\zeta \in \Theta'}$ . If  $\delta = \{\delta_w\}_{w \in \Theta}$  is a choice of weights for  $\dot{\mathbf{D}}_\tau$ , there is an induced set of weights for  $\varphi_\tau^* \dot{\mathbf{D}}_\tau$  defined by

$$\varphi^* \delta := \{k_\zeta \delta_{\varphi(\zeta)}\}_{\zeta \in \Theta'},$$

where  $k_\zeta \in \{1, \dots, d\}$  again denotes the branching order of  $\varphi$  at  $\zeta$ .

**Proposition 3.15.** *Suppose  $k \in \mathbb{N}$ ,  $p \in (1, \infty)$ , and the exponential weights  $\delta = \{\delta_w\}_{w \in \Theta}$  are chosen to satisfy*

$$0 < \delta_w < \frac{2\pi}{d}$$

*for every  $w \in \Theta$ . Then for any  $\mathbf{D}_\tau \in \mathcal{CR}_\mathbb{R}(E_\tau)$ , the operators*

$$\begin{aligned} \dot{\mathbf{D}}_\tau : W^{k,p,-\delta}(\dot{E}_\tau) &\rightarrow W^{k-1,p,-\delta}(\dot{F}_\tau), \\ \varphi_\tau^* \dot{\mathbf{D}}_\tau : W^{k,p,-\varphi^* \delta}(\varphi_\tau^* \dot{E}_\tau) &\rightarrow W^{k-1,p,-\varphi^* \delta}(\varphi_\tau^* \dot{F}_\tau) \end{aligned}$$

*are Fredholm and satisfy*

$$\text{ind}(\dot{\mathbf{D}}_\tau) = \text{ind}(\mathbf{D}_\tau), \quad \text{and} \quad \text{ind}(\varphi_\tau^* \dot{\mathbf{D}}_\tau) = \text{ind}(\varphi_\tau^* \mathbf{D}_\tau).$$

*Moreover, the maps  $\Gamma(E_\tau) \rightarrow \Gamma(\dot{E}_\tau)$  and  $\Gamma(\varphi_\tau^* E_\tau) \rightarrow \Gamma(\varphi_\tau^* \dot{E}_\tau)$  defined by restricting smooth sections to the corresponding punctured domains define isomorphisms*

$$\ker \mathbf{D}_\tau \xrightarrow{\cong} \ker \dot{\mathbf{D}}_\tau \quad \text{and} \quad \ker(\varphi_\tau^* \mathbf{D}_\tau) \xrightarrow{\cong} \ker(\varphi_\tau^* \dot{\mathbf{D}}_\tau).$$

*Proof.* We will prove the correspondence between  $\mathbf{D}_\tau$  and  $\dot{\mathbf{D}}_\tau$ , as the result for the pulled back operators follows by the same argument simply replacing the bundles  $E_\tau \rightarrow \Sigma$  and  $\dot{E}_\tau \rightarrow \dot{\Sigma}_\tau$  with  $\varphi_\tau^* E_\tau \rightarrow \Sigma'$  and  $\varphi_\tau^* \dot{E}_\tau \rightarrow \dot{\Sigma}'_\tau$  respectively.

The Fredholm property for  $\dot{\mathbf{D}}_\tau$  and the index calculation follow from the usual index formula for Cauchy-Riemann operators on Riemann surfaces with cylindrical ends, proved in [Sch95], supplemented by the transformation (3.4) to handle the exponential weights (cf. [HWZ99, §6]). In particular, the condition  $-2\pi < -\delta_w < 0$  for each  $w \in \Theta_\tau$  guarantees that  $\dot{\mathbf{D}}_\tau$  is conjugate (cf. (3.7) and (3.8) below) to a Cauchy-Riemann type operator  $W^{k,p}(\dot{E}_\tau) \rightarrow W^{k-1,p}(\dot{F}_\tau)$  with nondegenerate asymptotic operators at every puncture whose Conley-Zehnder indices with respect to the trivialization  $\Phi$  are  $m = \text{rank}_\mathbb{C} E_\tau$ . In light of (3.1), the index formula from [Sch95] thus gives

$$\text{ind}(\dot{\mathbf{D}}_\tau) = m\chi(\dot{\Sigma}_\tau) + 2c_1^\Phi(\dot{E}_\tau) + m \cdot |\Theta_\tau| = m\chi(\Sigma) + 2c_1(E_\tau) = \text{ind}(\mathbf{D}_\tau).$$

Note that doing the same computation for the pulled back operators requires the stronger condition  $-2\pi/d < -\delta_w < 0$  in order to ensure that all of the pulled back weights in the set  $-\varphi^*\delta$  lie in the interval  $(-2\pi, 0)$ .

To understand the kernels, observe that since any  $\eta \in \ker \mathbf{D}_\tau$  is smooth, its restriction to  $\dot{\Sigma}_\tau$  belongs to  $W^{k,p,-\delta}(\dot{E}_\tau)$  and is thus in  $\ker \dot{\mathbf{D}}_\tau$ .<sup>4</sup> Conversely, we need to show that any section  $\eta \in W^{k,p,-\delta}(\dot{E}_\tau)$  annihilated by  $\dot{\mathbf{D}}_\tau$  can be extended over the punctures to a section in  $W^{k,p}(E_\tau)$ , which is then automatically annihilated by  $\mathbf{D}_\tau$ . This will follow from the asymptotic elliptic theory of the equation  $\dot{\mathbf{D}}_\tau \eta = 0$ . Indeed, recall from (3.3) that on the cylindrical end near any puncture  $w \in \Theta_\tau$ , the function  $\eta(s, t) \in \mathbb{C}^m$  representing  $\eta \in \ker \dot{\mathbf{D}}_\tau$  in some trivialization satisfies

$$\bar{\partial}\eta + \dot{A}_\tau^{(w)}\eta \equiv 0,$$

and

$$\eta = e^{\delta s} f \quad \text{for some } f \in W^{k,p}([0, \infty) \times S^1, \mathbb{C}^m),$$

where  $\delta := \delta_w \in (0, 2\pi)$ . Then  $f = e^{-\delta s} \eta$  satisfies the Cauchy-Riemann type equation

$$(3.7) \quad \bar{\partial}f + (\delta + \dot{A}_\tau^{(w)})f = \partial_s f - [-i\partial_t - (\delta + \dot{A}_\tau^{(w)})]f = 0.$$

Since  $\dot{A}_\tau^{(w)}(s, \cdot) \rightarrow 0$  as  $s \rightarrow \infty$ , this equation is asymptotic to the equation  $(\partial_s - \mathbf{A}_\delta)f = 0$  for the asymptotic operator

$$(3.8) \quad \mathbf{A}_\delta := -i\partial_t - \delta : H^1(S^1, \mathbb{C}^m) \rightarrow L^2(S^1, \mathbb{C}^m),$$

which can be regarded as a densely defined unbounded self-adjoint operator on  $L^2(S^1, \mathbb{C}^m)$ . The function  $A_\tau^{(w)} : \mathbb{D} \rightarrow \text{End}_{\mathbb{R}}(\mathbb{C}^m)$  is smooth by assumption, and (3.2) then implies that the derivatives  $\partial^\alpha \dot{A}_\tau^{(w)}(s, t)$  of  $\dot{A}_\tau^{(w)}$  for arbitrary multi-indices  $\alpha$  satisfy exponential decay conditions

$$|\partial^\alpha \dot{A}_\tau^{(w)}(s, t)| \leq M_\alpha e^{-2\pi s}$$

for suitable constants  $M_\alpha > 0$ . Applying [Sie08, Theorem A.1],  $f$  therefore satisfies

$$f(s, t) = e^{\lambda s} [e(t) + r(s, t)],$$

where  $e : S^1 \rightarrow \mathbb{C}^m$  is a nontrivial eigenfunction of  $\mathbf{A}_\delta$  with eigenvalue  $\lambda < 0$ , and the remainder  $r(s, t) \in \mathbb{C}^m$  decays to zero with all its derivatives uniformly in  $t$  as  $s \rightarrow \infty$ . The spectrum of  $\mathbf{A}_\delta$  is  $\{2\pi k - \delta \mid k \in \mathbb{Z}\} \subset \mathbb{R}$ , hence the assumption  $\delta \in (0, 2\pi)$  implies  $\lambda \leq -\delta$ , and we conclude that

$$\eta(s, t) = e^{(\delta+\lambda)s} [e(t) + r(s, t)]$$

is bounded on the cylindrical end; in fact, one can use this to show that the smooth function  $\mathbb{D} \setminus \{0\} \rightarrow \mathbb{C}^m : z \mapsto \eta(z)$  defined via the transformation  $z = e^{-2\pi(s+it)}$  has finite  $W^{1,p}$ -norm on  $\mathbb{D} \setminus \{0\}$ . Moreover,  $\eta(z)$  has a continuous extension to  $z = 0$ : indeed, the extension is obviously  $\eta(0) = 0$  if  $\lambda < -\delta$ , while in the case  $\lambda = -\delta$ , the eigenfunction  $e(t)$  is necessarily constant, so that  $\eta(s, \cdot)$  converges to this constant value as  $s \rightarrow \infty$ . All these conditions together imply that the continuous extension of  $\eta$  over the punctures is of class

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<sup>4</sup>Note that  $\eta|_{\dot{\Sigma}_\tau}$  would not belong to  $W^{k,p,-\delta}(\dot{E}_\tau)$  in general if  $\eta$  were an arbitrary (not necessarily smooth) section of class  $W^{k,p}$  on  $E_\tau$ , nor if any of the exponential weights were nonnegative—the latter in particular permits sections in  $W^{k,p,-\delta}(\dot{E}_\tau)$  that do not decay to zero at infinity, which is crucial since arbitrary smooth sections  $\eta \in \ker \mathbf{D}_\tau$  may indeed be nonzero at points in  $\Theta_\tau$ .



$W^{k,p}$ , e.g. the case  $k = 1$  is a standard exercise using the definition of weak derivatives (cf. [Wen, Exercise 2.118]), and the general case follows from this by elliptic regularity.  $\square$

*Remark 3.16.* Since sections in  $W^{k,p,-\delta}(\dot{E}_\tau)$  and its pulled back counterpart need not be bounded when the weights  $-\delta$  are negative, the punctured operators in Proposition 3.15 cannot be interpreted in any reasonable way as linearizations of nonlinear Cauchy-Riemann operators, e.g.  $W^{k,p,-\delta}(\dot{E}_\tau)$  in this case is not a subspace of a tangent space in any reasonable Banach manifold. For our purposes, the exponential growth condition is merely a technical convenience so that we can consider operators with the right index and the right kernel and cokernel while dealing with honest covering maps instead of branched covers. The geometrically meaningful operators are still  $\mathbf{D}_\tau$  and  $\varphi_\tau^* \mathbf{D}_\tau$ , on unpunctured domains.

*Remark 3.17.* Suppose  $E_\tau$ ,  $\Sigma_\tau$  and  $\mathbf{D}_\tau$  are independent of  $\tau$  but  $\varphi_\tau$  moves in  $\mathcal{M}_\mathbf{b}^d(j)$  as  $\tau$  varies, e.g. this is the relevant situation for the proof of super-rigidity. There is then a subtle but important difference between what Proposition 3.15 says about  $\dot{\mathbf{D}}_\tau$  and what it says about  $\varphi_\tau^* \dot{\mathbf{D}}_\tau$ . The former is a family of operators whose relationship to each other for different values of  $\tau$  is not obvious from the definitions, but the proposition implies that they are all in some sense equivalent to a single operator  $\mathbf{D}$  on the closed domain, so they all have isomorphic kernels. No such thing can be assumed for the pulled back operators: while  $\varphi_\tau^* \dot{\mathbf{D}}_\tau$  must have the same index for all  $\tau$ , there is nothing in this setup to stop the dimension of its kernel from varying wildly with  $\tau$ .

**3.3. A digression on representation theory.** In preparation for the twisted bundle construction in the next section, we now collect some general facts from representation theory.

**3.3.1. Real permutation representations and subrepresentations.** Given a finite set  $I$  with  $d := |I| \in \mathbb{N}$  elements and a finite group with a homomorphism

$$\rho : G \rightarrow S(I) : g \mapsto \rho_g$$

defining a transitive group action on  $I$ , we denote by  $\mathbb{R}^I$  the real vector space spanned by basis vectors  $\{e_i\}_{i \in I}$ , with an inner product such that this basis is orthonormal. We shall use the boldface symbol  $\boldsymbol{\rho}$  to denote the corresponding real  $d$ -dimensional representation of  $G$ ,

$$(3.9) \quad \boldsymbol{\rho} : G \rightarrow \text{Aut}_{\mathbb{R}}(\mathbb{R}^I) \quad \text{such that} \quad \boldsymbol{\rho}(g)e_i := e_{\rho_g(i)}.$$

We will be interested in the decomposition of  $\mathbb{R}^I$  into irreducible  $G$ -invariant summands. This is easiest to understand in terms of its complexification

$$\boldsymbol{\rho}_{\mathbb{C}} : G \rightarrow \text{Aut}_{\mathbb{C}}(\mathbb{C}^I),$$

defined by viewing  $\{e_i\}_{i \in I}$  as a complex basis of  $\mathbb{C}^I$ . In general, we say that a complex representation  $\boldsymbol{\lambda} : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$  is the **complexification** of a real representation  $\boldsymbol{\theta} : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$  if  $V$  is isomorphic to  $W \oplus iW$  such that  $G$  acts on the latter by the complex-linear extension of its action on  $W$ . Recall from [Ser77, §13.2] that irreducible complex representations  $\boldsymbol{\lambda} : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$  come in three mutually exclusive types:

- **Real type:**  $V$  admits a complex-antilinear  $G$ -invariant involution. Then  $\lambda$  is the complexification of a real irreducible representation  $\theta : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$ . It follows that  $\lambda$  is isomorphic to its dual representation  $\lambda^* : G \rightarrow \text{Aut}_{\mathbb{C}}(V^*)$ , and all  $G$ -equivariant linear maps  $W \rightarrow W$  are given by scalar multiplication:

$$\text{End}_G(W) \cong \mathbb{R}.$$

- **Complex type:**  $\lambda$  is not isomorphic to its dual representation  $\lambda^* : G \rightarrow \text{Aut}_{\mathbb{C}}(V^*)$ . Then  $\lambda \oplus \lambda^* : G \rightarrow \text{Aut}_{\mathbb{C}}(V \oplus V^*)$  is the complexification of a real irreducible representation  $\theta : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$  obtained from  $\lambda : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$  by setting  $W := V$  and using the obvious inclusion  $\text{Aut}_{\mathbb{C}}(V) \subset \text{Aut}_{\mathbb{R}}(W)$ . The space of  $G$ -equivariant real-linear maps on  $W$  is then real 2-dimensional:

$$\text{End}_G(W) \cong \mathbb{C}.$$

- **Quaternionic type:**  $\lambda$  is not of real type but is nonetheless isomorphic to its dual representation. Then  $\lambda \oplus \lambda : G \rightarrow \text{Aut}_{\mathbb{C}}(V \oplus V)$  is the complexification of a real irreducible representation  $\theta : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$  obtained from  $\lambda : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$  by setting  $W := V$  and using the obvious inclusion  $\text{Aut}_{\mathbb{C}}(V) \subset \text{Aut}_{\mathbb{R}}(W)$ , and the space of  $G$ -equivariant real-linear maps on  $W$  is real 4-dimensional, i.e. it is isomorphic to the quaternions:

$$\text{End}_G(W) \cong \mathbb{H}.$$

We shall also refer to a real irreducible representation as “of **real/complex/quaternionic type**” according to which of these three constructions it comes from. With this classification in mind, we denote the various complex irreducible representations of  $G$  by

$$\lambda_{j,\mathbb{K}} : G \rightarrow \text{Aut}_{\mathbb{C}}(V_{j,\mathbb{K}}),$$

where  $\mathbb{K}$  stands for  $\mathbb{R}$ ,  $\mathbb{C}$  or  $\mathbb{H}$  depending on the type, and arrange a complete list of pairwise non-isomorphic irreducible representations in the form

$$\lambda_{1,\mathbb{R}}, \dots, \lambda_{p,\mathbb{R}}, \lambda_{1,\mathbb{C}}, \lambda_{1,\mathbb{C}}^*, \dots, \lambda_{q,\mathbb{C}}, \lambda_{q,\mathbb{C}}^*, \lambda_{1,\mathbb{H}}, \dots, \lambda_{n,\mathbb{H}}.$$

This gives rise to a corresponding complete list

$$\theta_{1,\mathbb{R}}, \dots, \theta_{p,\mathbb{R}}, \theta_{1,\mathbb{C}}, \dots, \theta_{q,\mathbb{C}}, \theta_{1,\mathbb{H}}, \dots, \theta_{n,\mathbb{H}}$$

of pairwise non-isomorphic real irreducible representations

$$\theta_{j,\mathbb{K}} : G \rightarrow \text{Aut}_{\mathbb{R}}(W_{j,\mathbb{K}}) \quad \text{satisfying} \quad \text{End}_G(W_{j,\mathbb{K}}) \cong \mathbb{K},$$

where for each  $j$ , the complexification of  $\theta_{j,\mathbb{K}}$  is  $\lambda_{j,\mathbb{K}}$  for  $\mathbb{K} = \mathbb{R}$ ,  $\lambda_{j,\mathbb{C}} \oplus \lambda_{j,\mathbb{C}}^*$  for  $\mathbb{K} = \mathbb{C}$ , and  $\lambda_{j,\mathbb{H}} \oplus \lambda_{j,\mathbb{H}}$  for  $\mathbb{K} = \mathbb{H}$ .

Since  $\rho_{\mathbb{C}}$  itself is a complexification of a real representation, every subspace in the resulting isotypic decomposition of  $\mathbb{C}^I$  is either identical or orthogonal to its complex conjugate, where the conjugate always carries the dual representation. Thus we can uniquely decompose  $\mathbb{C}^I$  into pairwise orthogonal  $G$ -invariant complex subspaces

$$(3.10) \quad \mathbb{C}^I = X_{1,\mathbb{R}} \oplus \dots \oplus X_{p,\mathbb{R}} \oplus X_{1,\mathbb{C}} \oplus \overline{X}_{1,\mathbb{C}} \oplus \dots \oplus X_{q,\mathbb{C}} \oplus \overline{X}_{q,\mathbb{C}} \oplus X_{1,\mathbb{H}} \oplus \dots \oplus X_{n,\mathbb{H}},$$

where each  $X_{j,\mathbb{R}}$  and  $X_{j,\mathbb{H}}$  is of the form  $Y_{j,\mathbb{K}} \oplus iY_{j,\mathbb{K}}$  for some real subspace  $Y_{j,\mathbb{K}} \subset \mathbb{R}^I$ , and each  $X_{j,\mathbb{C}}$  has trivial intersection with  $\mathbb{R}^I$ . Next, observe that every irreducible  $G$ -invariant subspace in  $\mathbb{C}^I$  is either identical to its complex conjugate or intersects it trivially:

indeed, any other option would produce an intersection which is a nontrivial but smaller  $G$ -invariant subspace. We can thus further decompose  $X_{j,\mathbb{R}}$  and  $X_{j,\mathbb{C}}$  into irreducible  $G$ -invariant subspaces

$$X_{j,\mathbb{R}} \cong V_{j,\mathbb{R}}^{\oplus k_j}, \quad X_{j,\mathbb{C}} \cong V_{j,\mathbb{C}}^{\oplus m_j}$$

for some integers  $k_j, m_j \geq 0$ , where each  $V_{j,\mathbb{R}}$  summand in  $X_{j,\mathbb{R}}$  can be assumed of the form  $W_{j,\mathbb{R}} \oplus iW_{j,\mathbb{R}}$  for some irreducible  $G$ -invariant real subspace  $W_{j,\mathbb{R}} \subset Y_{j,\mathbb{R}}$ . In  $X_{j,\mathbb{H}}$ , the irreducible  $G$ -invariant subspaces cannot be complexifications since the corresponding representation is not realizable over  $\mathbb{R}$ , thus these subspaces have trivial intersection with  $\mathbb{R}^I$  and can instead be arranged in conjugate pairs:

$$X_{j,\mathbb{H}} \cong V_{j,\mathbb{H}}^{\oplus \ell_j} \oplus \overline{V_{j,\mathbb{H}}^{\oplus \ell_j}}$$

for some integers  $\ell_j \geq 0$ . From this decomposition of  $\rho_{\mathbb{C}}$  we can immediately read off a corresponding decomposition of  $\rho$ : we have

$$(3.11) \quad \mathbb{R}^I = Y_{1,\mathbb{R}} \oplus \dots \oplus Y_{p,\mathbb{R}} \oplus Y_{1,\mathbb{C}} \oplus \dots \oplus Y_{q,\mathbb{C}} \oplus Y_{1,\mathbb{H}} \oplus \dots \oplus Y_{n,\mathbb{H}},$$

where the summands are all  $G$ -invariant and pairwise orthogonal,  $Y_{j,\mathbb{K}} = X_{j,\mathbb{K}} \cap \mathbb{R}^I$  for  $\mathbb{K} = \mathbb{R}, \mathbb{H}$ , and  $Y_{j,\mathbb{C}} = (X_{j,\mathbb{C}} \oplus \overline{X_{j,\mathbb{C}}}) \cap \mathbb{R}^I$ . These summands then admit further (non-unique) decompositions into real irreducible  $G$ -invariant subspaces

$$Y_{j,\mathbb{R}} \cong W_{j,\mathbb{R}}^{\oplus k_j}, \quad Y_{j,\mathbb{C}} \cong W_{j,\mathbb{C}}^{\oplus m_j}, \quad Y_{j,\mathbb{H}} \cong W_{j,\mathbb{H}}^{\oplus \ell_j}.$$

**3.3.2. The regular case.** We now specialize the above discussion to the case

$$I := G, \quad \rho_g(h) := gh,$$

in which case  $\rho : G \rightarrow \text{Aut}_{\mathbb{R}}(\mathbb{R}^G)$  is the so-called **regular representation** of  $G$ . By a standard theorem in complex representation theory, the complexification  $\rho_{\mathbb{C}} : G \rightarrow \text{Aut}_{\mathbb{C}}(\mathbb{C}^G)$  then contains every irreducible complex representation  $\lambda_{j,\mathbb{K}} : G \rightarrow \text{Aut}_{\mathbb{C}}(V_{j,\mathbb{K}})$  as a subrepresentation with multiplicity equal to  $\dim_{\mathbb{C}} V_{j,\mathbb{K}}$ . This implies a similar fact about  $\rho$  that we will make use of in §6 for proving Theorem D:

**Lemma 3.18.** *The real regular representation  $\rho : G \rightarrow \text{Aut}_{\mathbb{R}}(\mathbb{R}^G)$  contains every irreducible real representation  $\theta_{j,\mathbb{K}} : G \rightarrow \text{Aut}_{\mathbb{R}}(W_{j,\mathbb{K}})$  of  $G$  as a subrepresentation with positive multiplicity.  $\square$*

Next, recall that the action of  $G$  on itself by right multiplication

$$G \rightarrow S(G) : g \mapsto \rho'_g, \quad \rho'_g h := hg^{-1}$$

commutes with  $\rho$  and thus defines a second permutation representation

$$\rho' : G \rightarrow \text{Aut}_{\mathbb{R}}(\mathbb{R}^G), \quad \rho'(g)e_h = e_{hg^{-1}}$$

which commutes with  $\rho$ , giving rise to a representation

$$(3.12) \quad G \times G \rightarrow \text{Aut}_{\mathbb{R}}(\mathbb{R}^G) : (g, h) \mapsto \rho(g)\rho'(h).$$

By another standard theorem of complex representation theory, the summands in the isotypic decomposition (3.10) of  $\mathbb{C}^G$  are then invariant under the complexification of the  $(G \times G)$ -action (3.12), and they define irreducible complex representations of  $G \times G$ . In particular,  $\rho'$  therefore preserves each isotypic component for  $\rho$  but does not preserve

any further decomposition of that component into irreducible  $G$ -invariant subspaces. For future use, we note one additional fact from complex representation theory: the action of  $G \times G$  on an isotypic component in  $\mathbb{C}^G$  corresponding to a given irreducible representation  $\lambda : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$  is isomorphic to  $V \otimes V^*$ , with  $G \times G$  acting by

$$(G \times H) \times (V \otimes V^*) \rightarrow V \otimes V^* : ((g, h), v \otimes \alpha) \mapsto \lambda(g)v \otimes \lambda^*(h)\alpha,$$

cf. [Ser77, §6.2].

**3.3.3. Non-faithful representations.** An important special case of the factorization construction in Example 3.8 arises when

$$\theta : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$$

is a real irreducible representation that is not faithful. Choosing  $H$  to be any nontrivial normal subgroup of its kernel

$$H \subset \ker \theta \subset G,$$

$G/H$  then inherits an irreducible representation

$$\theta_H : G/H \rightarrow \text{Aut}_{\mathbb{R}}(W).$$

For example one can take  $H = \ker \theta$ , in which case  $\theta_H$  becomes faithful. Now if  $\rho : G \rightarrow S(I)$  is a transitive action on the set  $I$  of  $d$  elements, let

$$\rho_H : G/H \rightarrow S(I/H)$$

denote the induced action on the set  $I/H$  of  $H$ -orbits, and consider the corresponding permutation representations

$$\rho : G \rightarrow \text{Aut}_{\mathbb{R}}(\mathbb{R}^I), \quad \rho_H : G/H \rightarrow \text{Aut}_{\mathbb{R}}(\mathbb{R}^{I/H}).$$

**Lemma 3.19.** *Under the assumptions described above, the multiplicity of  $\theta : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$  as a subrepresentation of  $\rho : G \rightarrow \text{Aut}_{\mathbb{R}}(\mathbb{R}^I)$  matches the multiplicity of  $\theta_H : G/H \rightarrow \text{Aut}_{\mathbb{R}}(W)$  as a subrepresentation of  $\rho_H : G/H \rightarrow \text{Aut}_{\mathbb{R}}(\mathbb{R}^{I/H})$ .*

*Proof.* Observe that in terms of the real/complex/quaternionic distinction described in §3.3.1,  $\theta$  and  $\theta_H$  are necessarily of the same type: indeed, the spaces of linear maps on  $W$  that are  $G$ -equivariant or  $(G/H)$ -equivariant are the same since  $H$  acts trivially on  $W$ . The multiplicities of both are therefore determined in the same way by the multiplicities of the corresponding *complex* irreducible representations in the complexifications of  $\rho$  and  $\rho_H$  respectively, thus it will suffice to prove a similar statement about complex representations. Namely, assume  $\lambda : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$  is complex irreducible,  $H \subset \ker \lambda \subset G$  is a normal subgroup and  $\lambda_H : G/H \rightarrow \text{Aut}_{\mathbb{C}}(V)$  is the resulting irreducible representation of  $G/H$ . By orthonormality of characters, it will suffice to prove

$$\langle \chi_{\rho}, \chi_{\lambda} \rangle = \langle \chi_{\rho_H}, \chi_{\lambda_H} \rangle,$$

where the inner product of characters  $\chi_{\lambda} : G \rightarrow \mathbb{C}$  is given in general by

$$\langle \chi_{\lambda}, \chi_{\lambda'} \rangle := \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\lambda}(g)} \chi_{\lambda'}(g) \in \mathbb{C}.$$

For each  $i \in I$ , let  $G_i \subset G$  denote the stabilizer subgroup for  $i$  under the  $G$ -action on  $I$  via  $\rho$ . Since the action is transitive, the orbit-stabilizer theorem implies  $|G_i| = |G|/d$ . The

trace of a permutation matrix is the number of elements that it fixes, in other words the number of stabilizer subgroups that it belongs to, hence for each  $g \in G$ ,

$$\chi_{\rho}(g) = |\{i \in I \mid g \in G_i\}|.$$

This implies

$$(3.13) \quad \langle \chi_{\rho}, \chi_{\lambda} \rangle = \frac{1}{|G|} \sum_{i \in I} \sum_{g \in G_i} \chi_{\lambda}(g).$$

This can be simplified since  $G$  acts transitively on  $I$ , so the subgroups  $G_i$  for distinct  $i \in I$  are all conjugate. By the conjugation-invariance of characters, this implies that all  $d$  of the sums over  $G_i$  in (3.13) are identical, so plugging in  $|G_i| = |G|/d$ , we have

$$\langle \chi_{\rho}, \chi_{\lambda} \rangle = \frac{1}{|G_i|} \sum_{g \in G_i} \chi_{\lambda}(g),$$

where  $i \in I$  in this expression can be chosen arbitrarily.

To write down a similar expression for  $\langle \chi_{\rho_H}, \chi_{\lambda_H} \rangle$ , define for each  $i \in I$

$$H_i := H \cap G_i \subset G,$$

which is a subgroup of both  $H$  and  $G_i$  and is normal in the latter. There is then a natural inclusion of  $G_i/H_i$  as a subgroup of  $G/H$ , and it is the stabilizer subgroup of  $[i] \in I/H$  for the permutation action of  $G/H$  on  $I/H$ . The same computation thus gives

$$\langle \chi_{\rho_H}, \chi_{\lambda_H} \rangle = \frac{1}{|G_i/H_i|} \sum_{[g] \in G_i/H_i} \chi_{\lambda_H}([g]) = \frac{|H_i|}{|G_i|} \sum_{[g] \in G_i/H_i} \chi_{\lambda_H}([g]).$$

Finally, observe that  $\chi_{\lambda}(g) = \chi_{\lambda_H}([g])$  for each  $g \in G$  since both are traces of the same matrix acting on  $V$ , so one can replace the last expression with a sum over  $g \in G_i$ , giving

$$\langle \chi_{\rho_H}, \chi_{\lambda_H} \rangle = \frac{1}{|G_i|} \sum_{g \in G_i} \chi_{\lambda}(g) = \langle \chi_{\rho}, \chi_{\lambda} \rangle.$$

□

**3.4. Twisted bundles and splittings of operators.** We can now make precise the splitting of pulled back Cauchy-Riemann type operators that was sketched in §2.2.

**3.4.1. Twisted bundles from representations.** We associate to any real representation  $\theta : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$  the family of real vector bundles  $W_{\tau}^{\theta} \rightarrow \dot{\Sigma}_{\tau}$  defined by

$$W_{\tau}^{\theta} = \left( \dot{\Sigma}_{\tau}'' \times W \right) / G,$$

where  $G$  acts on  $W$  via  $\theta$  and on  $\dot{\Sigma}_{\tau}''$  by deck transformations, so that  $\pi_{\tau} : \dot{\Sigma}_{\tau}'' \rightarrow \dot{\Sigma}_{\tau}$  identifies the latter with  $\dot{\Sigma}_{\tau}''/G$ . This gives rise to complex vector bundles  $\dot{E}_{\tau}^{\theta}, \dot{F}_{\tau}^{\theta} \rightarrow \dot{\Sigma}_{\tau}$  of rank  $m \cdot \dim_{\mathbb{R}} W$ , defined by

$$\dot{E}_{\tau}^{\theta} = \dot{E}_{\tau} \otimes_{\mathbb{R}} W_{\tau}^{\theta}, \quad \dot{F}_{\tau}^{\theta} = \dot{F}_{\tau} \otimes_{\mathbb{R}} W_{\tau}^{\theta} = \overline{\text{Hom}_{\mathbb{C}}}(T\dot{\Sigma}_{\tau}, \dot{E}_{\tau}^{\theta}).$$

Each of the bundles  $W_\tau^\theta$  has a canonical flat structure, i.e. it comes with a well-defined notion of locally constant sections, thus  $\mathbf{D}_\tau \in \mathcal{CR}_\mathbb{R}(E_\tau)$  determines a family of Cauchy-Riemann type operators

$$\dot{\mathbf{D}}_\tau^\theta : \Gamma(\dot{E}_\tau^\theta) \rightarrow \Gamma(\dot{F}_\tau^\theta) = \Omega^{0,1}(\dot{\Sigma}_\tau, \dot{E}_\tau^\theta)$$

such that  $\dot{\mathbf{D}}_\tau^\theta(\eta \otimes v) = \dot{\mathbf{D}}_\tau \eta \otimes v$  whenever  $v$  is a locally constant section of  $W_\tau^\theta$ . Since  $\dot{\mathbf{D}}_\tau^\theta \in \mathcal{CR}_\mathbb{R}(\dot{E}_\tau^\theta)$ , it is Fredholm in suitable Banach space settings, in particular as a bounded linear operator

$$\dot{\mathbf{D}}_\tau^\theta : W^{k,p,-\delta}(\dot{E}_\tau^\theta) \rightarrow W^{k-1,p,-\delta}(\dot{F}_\tau^\theta)$$

for any  $k \in \mathbb{N}$ ,  $p \in (1, \infty)$ , and negative exponential weights  $-\delta = \{-\delta_w\}_{w \in \Theta}$  with all  $\delta_w > 0$  sufficiently small. We will formulate a precise version of this statement and compute the index in §4. Observe that aside from its obvious dependence on  $\mathbf{D}_\tau$ ,  $\dot{\mathbf{D}}_\tau^\theta$  depends on our choice of regular presentation for  $\varphi$  and on the representation  $\theta$ , but both of them only up to isomorphism.

The most important special case of the above construction is  $\dot{E}_\tau^\rho \rightarrow \dot{\Sigma}_\tau$ , where  $\rho : G \rightarrow \text{Aut}_\mathbb{R}(\mathbb{R}^I)$  is the permutation representation associated to our regular presentation of  $\varphi$ . We define  $\dot{E}_\tau^\rho = \dot{E}_\tau \otimes (\mathbb{R}^I)_\tau^\rho \rightarrow \dot{\Sigma}_\tau$  as above and can identify it canonically with

$$\dot{E}_\tau^\rho = \left( \pi_\tau^* \dot{E}_\tau \otimes \mathbb{R}^I \right) / G,$$

so that sections of  $\dot{E}_\tau^\rho$  can be written as  $G$ -equivariant sections of  $\pi_\tau^* \dot{E}_\tau \otimes \mathbb{R}^I$ , hence

$$\eta = \sum_{i \in I} \eta^i \otimes e_i$$

for  $\eta^i \in \Gamma(\pi_\tau^* \dot{E}_\tau)$ . Here  $G$ -equivariance means that for all  $z \in \dot{\Sigma}_\tau''$  and  $g \in G$ ,

$$\eta(gz) = (\mathbf{1} \otimes \rho(g))\eta(z) = \sum_{i \in I} \eta^i(z) \otimes e_{\rho_g(i)},$$

hence

$$(3.14) \quad \eta^i(z) = \eta^{\rho_g(i)}(gz) \quad \text{for all } z \in \dot{\Sigma}_\tau'', g \in G \text{ and } i \in I.$$

Writing  $\dot{\Sigma}_\tau = (\dot{\Sigma}_\tau'' \times I)/G$ , this relation gives rise to a bijective correspondence

$$(3.15) \quad \begin{aligned} \Gamma(\dot{E}_\tau^\rho) &\rightarrow \Gamma(\varphi_\tau^* \dot{E}_\tau) : \eta \mapsto \hat{\eta} \\ \hat{\eta}([(z, i)]) &= \eta^i(z) \end{aligned}$$

and thus natural isomorphisms

$$W^{k,p,-\delta}(\dot{E}_\tau^\rho) \rightarrow W^{k,p,-\delta'}(\varphi_\tau^* \dot{E}_\tau)$$

for every  $k \geq 0$  and  $p \in (1, \infty)$ , where the exponential weights  $\delta = \{\delta_w > 0\}_{w \in \Theta}$  and  $\delta' = \{\delta_z > 0\}_{z \in \Theta'}$  can all be assumed arbitrarily close to zero.

Observe that  $(\mathbb{R}^I)_\tau^\rho \rightarrow \dot{\Sigma}_\tau$  also has a well-defined real bundle metric since  $\rho$  acts on  $\mathbb{R}^I$  by orthogonal transformations, so endowing  $E_\tau$  with a Hermitian bundle metric induces a Hermitian bundle metric on  $\dot{E}_\tau^\rho = \dot{E}_\tau \otimes (\mathbb{R}^I)_\tau^\rho$  such that the correspondence (3.15) also preserves  $L^2$ -products. After writing down a similar correspondence for the bundles  $\dot{F}_\tau^\rho$

and  $\varphi_\tau^* \dot{F}_\tau$ , we obtain an identification between the Cauchy-Riemann operators  $\varphi_\tau \dot{\mathbf{D}}_\tau$  and  $\dot{\mathbf{D}}_\tau^\rho$ :

$$(3.16) \quad \begin{array}{ccc} W^{k,p,-\delta}(\dot{E}_\tau^\rho) & \xrightarrow{\dot{\mathbf{D}}_\tau^\rho} & W^{k-1,p,-\delta}(\dot{F}_\tau^\rho) \\ \downarrow \cong & & \downarrow \cong \\ W^{k,p,-\delta'}(\varphi_\tau^* \dot{E}_\tau) & \xrightarrow{\varphi_\tau^* \dot{\mathbf{D}}_\tau} & W^{k-1,p,-\delta'}(\varphi_\tau^* \dot{F}_\tau), \end{array}$$

**3.4.2. Splitting the twisted Cauchy-Riemann operator.** If  $W \subset \mathbb{R}^I$  is any  $G$ -invariant subspace and  $\theta : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$  denotes the resulting subrepresentation, then we obtain corresponding subbundles

$$\dot{E}_\tau^\theta \subset \dot{E}_\tau^\rho, \quad \dot{F}_\tau^\theta \subset \dot{F}_\tau^\rho$$

such that  $\dot{\mathbf{D}}_\tau^\rho$  takes sections of  $\dot{E}_\tau^\theta$  to sections of  $\dot{F}_\tau^\theta$ , acting as the operator  $\dot{\mathbf{D}}_\tau^\theta$ . Under the correspondence (3.15), one can understand this as identifying  $\Gamma(\dot{E}_\tau^\theta)$  and  $\Gamma(\dot{F}_\tau^\theta)$  with closed subspaces

$$\Gamma_\theta(\varphi_\tau^* \dot{E}_\tau) \subset \Gamma(\varphi_\tau^* \dot{E}_\tau), \quad \Gamma_\theta(\varphi_\tau^* \dot{F}_\tau) \subset \Gamma(\varphi_\tau^* \dot{F}_\tau),$$

with a similar definition for closed subspaces of the relevant weighted Sobolev spaces, such that  $\varphi_\tau^* \dot{\mathbf{D}}_\tau$  restricts to a bounded linear operator

$$W_\theta^{k,p,-\delta'}(\varphi_\tau^* \dot{E}_\tau) \xrightarrow{\varphi_\tau^* \dot{\mathbf{D}}_\tau} W_\theta^{k-1,p,-\delta'}(\varphi_\tau^* \dot{F}_\tau),$$

which is conjugate to  $\dot{\mathbf{D}}_\tau^\theta : W^{k,p,-\delta}(\dot{E}_\tau^\theta) \rightarrow W^{k-1,p,-\delta}(\dot{F}_\tau^\theta)$  and will thus be Fredholm with any negative exponential weights that are close enough to 0. Now if

$$\mathbb{R}^I = W_1 \oplus \dots \oplus W_N$$

is a decomposition of  $\rho$  into subrepresentations  $\theta_j : G \rightarrow \text{Aut}_{\mathbb{R}}(W_j)$  for  $j = 1, \dots, N$ , we obtain a direct sum decomposition

$$\dot{\mathbf{D}}_\tau^\rho = \dot{\mathbf{D}}_\tau^{\theta_1} \oplus \dots \oplus \dot{\mathbf{D}}_\tau^{\theta_N},$$

which is equivalent via (3.16) to a decomposition of  $\varphi_\tau^* \dot{\mathbf{D}}_\tau$  over a splitting of Banach spaces

$$W^{k,p,-\delta'}(\varphi_\tau^* E_\tau) = \bigoplus_{j=1}^N W_{\theta_j}^{k,p,-\delta'}(\varphi_\tau^* E_\tau)$$

and the corresponding decomposition of  $W^{k-1,p,-\delta'}(\varphi_\tau^* F_\tau)$ . Note that if the subspaces  $W_1, \dots, W_N \subset \mathbb{R}^I$  are pairwise orthogonal, then the corresponding spaces of sections of  $\varphi_\tau^* \dot{E}_\tau$  and  $\varphi_\tau^* \dot{F}_\tau$  are  $L^2$ -orthogonal as a consequence. It is useful to observe that whenever two of the representations  $\theta_i : G \rightarrow \text{Aut}_{\mathbb{R}}(W_i)$  and  $\theta_j : G \rightarrow \text{Aut}_{\mathbb{R}}(W_j)$  are isomorphic, the  $G$ -equivariant isomorphism  $W_i \rightarrow W_j$  induces bundle isomorphisms  $\dot{E}_\tau^{\theta_i} \rightarrow \dot{E}_\tau^{\theta_j}$  and  $\dot{F}_\tau^{\theta_i} \rightarrow \dot{F}_\tau^{\theta_j}$  that identify  $\dot{\mathbf{D}}_\tau^{\theta_i}$  with  $\dot{\mathbf{D}}_\tau^{\theta_j}$ , so these two operators have isomorphic kernels and cokernels. This implies:



**Lemma 3.20.** *Suppose  $\theta_j : G \rightarrow \text{Aut}_{\mathbb{R}}(W_j)$  for  $j = 1, \dots, N$  is a set of pairwise non-isomorphic real representations of  $G$ , and  $\theta : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$  is another representation such that*

$$\theta \cong \bigoplus_{j=1}^N \theta_j^{\oplus k_j}$$

*for some integers  $k_1, \dots, k_N \geq 0$ . Then there exist isomorphisms*

$$\ker \dot{\mathbf{D}}_{\tau}^{\theta} \cong \bigoplus_{j=1}^N \left( \ker \dot{\mathbf{D}}_{\tau}^{\theta_j} \right)^{\oplus k_j} \quad \text{and} \quad \text{coker } \dot{\mathbf{D}}_{\tau}^{\theta} \cong \bigoplus_{j=1}^N \left( \text{coker } \dot{\mathbf{D}}_{\tau}^{\theta_j} \right)^{\oplus k_j}.$$

*In particular, if  $\theta$  is the permutation representation  $\rho : G \rightarrow \text{Aut}_{\mathbb{R}}(\mathbb{R}^I)$ , this gives isomorphisms*

$$\ker(\varphi_{\tau}^* \dot{\mathbf{D}}_{\tau}) \cong \bigoplus_{j=1}^N \left( \ker \dot{\mathbf{D}}_{\tau}^{\theta_j} \right)^{\oplus k_j} \quad \text{and} \quad \text{coker}(\varphi_{\tau}^* \dot{\mathbf{D}}_{\tau}) \cong \bigoplus_{j=1}^N \left( \text{coker } \dot{\mathbf{D}}_{\tau}^{\theta_j} \right)^{\oplus k_j}.$$

□

**3.4.3. Non-faithful representations revisited.** Here is a quick proof of Lemma 2.11. For the present discussion we drop the parameter  $\tau$  from the notation since it does not play any important role.

Suppose  $\theta : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$  is a representation and  $H \subset \ker \theta \subset G$  is a nontrivial normal subgroup, giving rise to a representation

$$\theta_H : G/H \rightarrow \text{Aut}_{\mathbb{R}}(W),$$

and (following Example 3.8) a factorization of  $\varphi : \Sigma' \rightarrow \Sigma$  as

$$\Sigma' \rightarrow \Sigma'_H \xrightarrow{\varphi_H} \Sigma.$$

By assumption we are using a minimal regular presentation and thus  $\rho : G \rightarrow S(I)$  is injective, so  $H$  acts nontrivially on  $I$ , implying  $\deg(\varphi_H) < d$ . Writing  $\dot{\Sigma}''_H = \dot{\Sigma}''/H$ , the obvious projection map

$$\left( \dot{\Sigma}'' \times W \right) / G \rightarrow \left( \dot{\Sigma}''_H \times W \right) / (G/H)$$

is then an isomorphism of real vector bundles over  $\dot{\Sigma}$  and thus gives rise to a canonical identification between the twisted bundles  $\dot{E}^{\theta}$  and  $\dot{E}^{\theta_H}$  with their Cauchy-Riemann operators  $\dot{\mathbf{D}}^{\theta}$  and  $\dot{\mathbf{D}}^{\theta_H}$ . To prove the lemma, we now just need to observe that Lemma 3.19 implies  $\theta$  is a subrepresentation of  $\rho$  if and only if  $\theta_H$  is a subrepresentation of  $\rho_H$ , hence the corresponding twisted operators appear simultaneously as summands in the decompositions of  $\varphi^* \dot{\mathbf{D}}$  and  $\varphi_H^* \dot{\mathbf{D}}$  from Lemma 3.20.

*Remark 3.21.* In the situation above, one should interpret  $\ker \dot{\mathbf{D}}^{\theta}$  as the set of all sections in  $\ker(\varphi^* \dot{\mathbf{D}})$  that are pullbacks of sections in  $\ker \dot{\mathbf{D}}^{\theta_H}$  (interpreted as a subspace of  $\ker(\varphi_H^* \dot{\mathbf{D}})$ ) via the branched cover  $\Sigma' \rightarrow \Sigma'_H$ .

3.4.4. *The regular case revisited.* Now consider the special case where  $\rho$  is the regular representation  $G \rightarrow \text{Aut}_{\mathbb{R}}(\mathbb{R}^G)$ , defined via

$$\rho : G \rightarrow S(G), \quad \rho_g(h) = gh.$$

We saw in Example 3.7 that this means  $\varphi_\tau : \dot{\Sigma}'_\tau \rightarrow \dot{\Sigma}_\tau$  are all regular covers isomorphic to  $\pi : \dot{\Sigma}''_\tau \rightarrow \dot{\Sigma}_\tau$ , and the action of  $G$  on  $\dot{\Sigma}'_\tau = (\dot{\Sigma}''_\tau \times G)/G$  by deck transformations takes the form

$$g[(z, h)] := [(z, \rho'_g(h))]$$

where  $\rho' : G \rightarrow S(G)$  is the action of  $G$  on itself by right multiplication,  $\rho'_g(h) = hg^{-1}$ . The induced  $G$ -action on spaces of sections  $\eta$  of  $\varphi_\tau^* \dot{E}_\tau$  is defined by

$$(g\eta)([(z, h)]) := \eta(g^{-1}[(z, h)]) = \eta([(z, hg)]).$$

Recall now from §3.3.2 that the permutation representation  $\rho' : G \rightarrow \text{Aut}_{\mathbb{R}}(\mathbb{R}^G)$  arising from  $\rho'$  commutes with  $\rho$  and preserves the isotypic components of  $\rho$ . It therefore defines an action on  $\dot{E}_\tau^\rho$  by fiber-preserving bundle isomorphisms, and these isomorphisms preserve each of the subbundles in the splitting

$$(3.17) \quad \dot{E}_\tau^\rho = \bigoplus_{j=1}^p (\dot{E}_\tau^\rho)_{j, \mathbb{R}} \oplus \bigoplus_{j=1}^q (\dot{E}_\tau^\rho)_{j, \mathbb{C}} \oplus \bigoplus_{j=1}^n (\dot{E}_\tau^\rho)_{j, \mathbb{H}}$$

corresponding to the isotypic decomposition (3.11) of  $\rho$ . In particular, this  $G$ -action by bundle isomorphisms gives a linear  $G$ -action on each of the subspaces  $\Gamma((\dot{E}_\tau^\rho)_{j, \mathbb{R}}) \subset \Gamma(\dot{E}_\tau^\rho)$ , and there is a similar action on sections of  $\dot{E}_\tau^\rho$  such that the restriction of  $\dot{\mathbf{D}}_\tau^\rho$  to each of these subspaces is  $G$ -equivariant. Its kernel and cokernel thus inherit natural  $G$ -actions. Under the correspondence (3.15), this action on sections of  $\dot{E}_\tau^\rho$  matches the action by deck transformations on  $\Gamma(\varphi_\tau^* \dot{E}_\tau)$ .

**Lemma 3.22.** *Suppose  $\rho : G \rightarrow S(G)$  is defined by left multiplication,  $\theta_0 : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$  is a real irreducible representation of  $G$ , and  $\theta : G \rightarrow \text{Aut}_{\mathbb{R}}(Y)$  denotes the corresponding summand in the isotypic decomposition (3.11) of the regular representation  $\rho : G \rightarrow \text{Aut}_{\mathbb{R}}(\mathbb{R}^G)$ . Then every irreducible subrepresentation for the natural  $G$ -action on  $\ker \dot{\mathbf{D}}_\tau^\theta$  or  $\text{coker } \dot{\mathbf{D}}_\tau^\theta$  is isomorphic to  $\theta_0$ .*

*Proof.* Suppose first that  $\theta_0$  is of either real or quaternionic type, in which case the complexification  $X := Y \oplus iY \subset \mathbb{C}^G$  of  $Y \subset \mathbb{R}^G$  is also an isotypic component for the complexified regular representation  $\rho_{\mathbb{C}} : G \rightarrow \text{Aut}_{\mathbb{C}}(\mathbb{C}^G)$ . We shall denote the restriction of  $\rho_{\mathbb{C}}$  to  $X$  by

$$\lambda : G \rightarrow \text{Aut}_{\mathbb{C}}(X),$$

and let  $\lambda_0 : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$  denote the underlying complex irreducible representation. Regarding these complex representations as real representations on  $X$  and  $V$  respectively gives rise to corresponding twisted bundles and Cauchy-Riemann operators on them, along with a natural linear inclusion of vector bundles

$$\dot{E}_\tau^\theta \hookrightarrow \dot{E}_\tau^\lambda \quad \text{such that} \quad \ker \dot{\mathbf{D}}_\tau^\theta = \ker \dot{\mathbf{D}}_\tau^\lambda \cap \Gamma(\dot{E}_\tau^\theta).$$

It will be useful to think of  $\dot{E}_\tau^\lambda$  as a *complexification* of  $\dot{E}_\tau^\theta$ , in the following sense. While  $\dot{E}_\tau^\theta$  is already a complex vector bundle,  $\dot{E}_\tau^\lambda = \dot{E}_\tau \otimes_{\mathbb{R}} X_\tau^\lambda$  naturally carries *two* complex

structures  $J_\tau$  and  $i$ , which commute with each other: the former acts on  $\eta \otimes v \in \dot{E}_\tau \otimes_{\mathbb{R}} X_\tau^\lambda$  by  $J_\tau \eta \otimes v$  and the latter by  $\eta \otimes iv$ , using the fact that  $\lambda$  is a complex representation and  $X_\tau^\lambda$  is therefore naturally a complex vector bundle. From this perspective,  $\dot{D}_\tau^\lambda$  is the natural  $i$ -complex-linear extension of  $\dot{D}_\tau^\theta$  to its complexified domain, and the representations defined by the  $G$ -action on  $\ker \dot{D}_\tau^\lambda$  and  $\operatorname{coker} \dot{D}_\tau^\lambda$  will be the complexifications of the real representations it defines on  $\ker \dot{D}_\tau^\theta$  and  $\operatorname{coker} \dot{D}_\tau^\theta$  respectively. In the following we shall use the symbol “ $\otimes_i$ ” to denote complex tensor products of vector spaces and bundles with  $i$  (instead of  $J_\tau$ ) as the complex structure.

Recall now that as an isotypic component of the complex regular representation,  $X$  admits a complex-linear isomorphism to  $V \otimes_i V^*$  such that for all  $g \in G$ ,  $\rho(g)$  acts on  $V \otimes_i V^*$  as  $\lambda_0 \otimes \mathbb{1}$ , while  $\rho'(g)$  acts as  $\mathbb{1} \otimes \lambda_0^*$ . The isomorphism  $X \rightarrow V \otimes_i V^*$  thus gives rise to  $i$ -complex bundle isomorphisms

$$\dot{E}_\tau^\lambda \rightarrow \dot{E}_\tau^{\lambda_0} \otimes_i V^*, \quad \dot{F}_\tau^\lambda \rightarrow \dot{F}_\tau^{\lambda_0} \otimes_i V^*,$$

where we are abusing notation to let  $V^*$  denote the trivial bundle over  $\dot{\Sigma}_\tau$  with fiber  $V^*$ , and this identifies  $\dot{D}_\tau^\lambda$  with  $\dot{D}_\tau^{\lambda_0} \otimes \mathbb{1}$ . We therefore have

$$\ker \dot{D}_\tau^\lambda \cong \ker \dot{D}_\tau^{\lambda_0} \otimes_i V^*, \quad \operatorname{coker} \dot{D}_\tau^\lambda \cong \operatorname{coker} \dot{D}_\tau^{\lambda_0} \otimes_i V^*,$$

with  $G$  acting on both by  $\mathbb{1} \otimes \lambda_0^*$ , hence all irreducible subrepresentations in these spaces are isomorphic to  $\lambda_0^*$ , which is isomorphic to  $\lambda_0$  since the latter is not of complex type. Viewing these as complexifications of real representations on  $\ker \dot{D}_\tau^\theta$  and  $\operatorname{coker} \dot{D}_\tau^\theta$  as explained above, it follows via the correspondence between real and complex irreducible representations outlined in §3.3.1 that all the irreducible real subrepresentations are isomorphic to  $\theta_0$ .

The main difference if  $\theta_0$  is of complex type is that  $Y \oplus iY \subset \mathbb{C}^G$  is no longer an isotypic component for  $\rho_{\mathbb{C}}$ , but is instead the direct sum of two isotypic components related to each other by complex conjugation

$$Y \oplus iY = X \oplus \overline{X} \subset \mathbb{C}^G,$$

corresponding to some complex irreducible representation  $\lambda_0 : G \rightarrow \operatorname{Aut}_{\mathbb{C}}(V)$  and its non-isomorphic dual  $\lambda_0^* : G \rightarrow \operatorname{Aut}_{\mathbb{C}}(V^*)$ . Writing  $\lambda : G \rightarrow \operatorname{Aut}_{\mathbb{C}}(X)$  and  $\bar{\lambda} : G \rightarrow \operatorname{Aut}_{\mathbb{C}}(\overline{X})$  for the restriction of  $\rho_{\mathbb{C}}$  to these subspaces, we can then think of  $\dot{D}_\tau^{\lambda \oplus \bar{\lambda}} = \dot{D}_\tau^\lambda \oplus \dot{D}_\tau^{\bar{\lambda}}$  as the complexification of  $\dot{D}_\tau^\theta$ . A repeat of the argument above using the isomorphisms  $X \cong V \otimes_i V^*$  and  $\overline{X} \cong V^* \otimes_i V$  then gives an  $i$ -complex-linear isomorphism

$$\ker \dot{D}_\tau^{\lambda \oplus \bar{\lambda}} \cong (\ker \dot{D}_\tau^{\lambda_0} \otimes_i V^*) \oplus (\ker \dot{D}_\tau^{\lambda_0^*} \otimes_i V),$$

with  $G$  acting via  $\mathbb{1} \otimes \lambda_0^*$  on the first summand and  $\mathbb{1} \otimes \lambda_0$  on the second, and a similar isomorphism for cokernels. It follows that every irreducible subrepresentation in either  $\ker \dot{D}_\tau^{\lambda \oplus \bar{\lambda}}$  or  $\operatorname{coker} \dot{D}_\tau^{\lambda \oplus \bar{\lambda}}$  is isomorphic to one of  $\lambda_0$  or  $\lambda_0^*$ , and the desired result for real subrepresentations again follows via the correspondence between real and complex representations in §3.3.1.  $\square$

Combining this with Lemma 3.20 and applying Schur's lemma now gives:

**Corollary 3.23.** *In the setting of Lemma 3.22, let  $K := \dim \ker \dot{\mathbf{D}}_\tau^{\theta_0}$  and  $C := \dim \operatorname{coker} \dot{\mathbf{D}}_\tau^{\theta_0}$ . Then the space of  $G$ -equivariant real-linear maps  $\ker \dot{\mathbf{D}}_\tau^\theta \rightarrow \operatorname{coker} \dot{\mathbf{D}}_\tau^\theta$  has real dimension*

$$\dim \operatorname{Hom}_G (\ker \dot{\mathbf{D}}_\tau^\theta, \operatorname{coker} \dot{\mathbf{D}}_\tau^\theta) = \begin{cases} KC & \text{if } \theta_0 \text{ is of real type,} \\ 2KC & \text{if } \theta_0 \text{ is of complex type,} \\ 4KC & \text{if } \theta_0 \text{ is of quaternionic type.} \end{cases}$$

□

**3.5. Setting up the implicit function theorem.** We assume throughout this section that  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$  is the minimal regular presentation of  $\varphi : \Sigma' \rightarrow \Sigma$ . Suppose

$$\theta_i : G \rightarrow \operatorname{Aut}_{\mathbb{R}}(W_i), \quad i = 1, \dots, N$$

is a complete list of pairwise non-isomorphic real irreducible representations for  $G$ . Any  $N$ -tuples of nonnegative integers  $\mathbf{k} = (k_1, \dots, k_N)$  and  $\mathbf{c} = (c_1, \dots, c_N)$  now determine subsets of the parameter space

$$P(\mathbf{k}, \mathbf{c}) := \left\{ \tau \in P \mid \dim \ker \dot{\mathbf{D}}_\tau^{\theta_i} = k_i \text{ and } \dim \operatorname{coker} \dot{\mathbf{D}}_\tau^{\theta_i} = c_i \text{ for all } i = 1, \dots, N \right\}.$$

Note that  $P(\mathbf{k}, \mathbf{c})$  is automatically empty unless  $k_i - c_i = \operatorname{ind} \dot{\mathbf{D}}_\tau^{\theta_i}$  for all  $i = 1, \dots, N$ , and these indices do not depend on the parameter  $\tau$ . Assuming this condition holds, we would now like to present  $P(\mathbf{k}, \mathbf{c})$  locally as the zero-set of a smooth map to a finite-dimensional vector space, and to compute its derivative in a special case.

We start by translating the conditions defining  $P(\mathbf{k}, \mathbf{c})$  into conditions on the pulled back operators  $\hat{\varphi}_\tau^* \dot{\mathbf{D}}_\tau$  for a suitable family of *regular* covers  $\hat{\varphi}_\tau : \hat{\Sigma}_\tau \rightarrow \dot{\Sigma}_\tau$  with  $\operatorname{Aut}(\hat{\varphi}_\tau) = G$ . This can be defined by replacing the homomorphism  $\rho : G \rightarrow S(I)$  with the action of  $G$  on itself by left multiplication, i.e. let

$$\hat{\rho} : G \rightarrow S(G) : g \mapsto \hat{\rho}_g, \quad \hat{\rho}_g(h) := gh,$$

so that  $(\Theta_\tau, \dot{\Sigma}'', \pi_\tau, G, \hat{\rho}, G, \operatorname{Id})$  becomes a minimal regular presentation for

$$\hat{\Sigma}_\tau := \left( \dot{\Sigma}_\tau'' \times G \right) / G \xrightarrow{\hat{\varphi}_\tau} \dot{\Sigma}_\tau : [(z, g)] \mapsto \pi_\tau(z),$$

or rather for the extension of this map to a branched cover of closed surfaces as provided by Lemma 3.1. In keeping with our usual notational convention,  $\hat{\Sigma}_\tau$  is a fixed smooth surface  $\hat{\Sigma}$  with a fixed  $G$ -action by deck transformations but a  $\tau$ -dependent family of conformal structures  $\hat{j}_\tau = \hat{\varphi}_\tau^* j_\tau$ , which are fixed on the cylindrical ends.

Denote the isotypic decomposition of the regular representation  $\hat{\rho} : G \rightarrow \operatorname{Aut}_{\mathbb{R}}(\mathbb{R}^G)$  by

$$\hat{\rho} = \bigoplus_{i=1}^N \hat{\theta}_i,$$

where  $\widehat{\theta}_i \cong \theta_i^{\oplus \ell_i}$  for some integers  $\ell_i$  which are strictly positive by Lemma 3.18. Then by Lemma 3.20,

$$\begin{aligned} \ker(\hat{\varphi}_\tau^* \dot{\mathbf{D}}_\tau) &\cong \bigoplus_{i=1}^N \ker \dot{\mathbf{D}}_\tau^{\widehat{\theta}_i} \cong \bigoplus_{i=1}^N \left( \ker \dot{\mathbf{D}}_\tau^{\theta_i} \right)^{\oplus \ell_i}, \\ \operatorname{coker}(\hat{\varphi}_\tau^* \dot{\mathbf{D}}_\tau) &\cong \bigoplus_{i=1}^N \operatorname{coker} \dot{\mathbf{D}}_\tau^{\widehat{\theta}_i} \cong \bigoplus_{i=1}^N \left( \operatorname{coker} \dot{\mathbf{D}}_\tau^{\theta_i} \right)^{\oplus \ell_i}, \end{aligned}$$

so  $\tau \in P(\mathbf{k}, \mathbf{c})$  implies

$$(3.18) \quad \dim \ker(\hat{\varphi}_\tau^* \dot{\mathbf{D}}_\tau) = \sum_{i=1}^N \ell_i k_i.$$

**Lemma 3.24.** *Every  $\sigma \in P(\mathbf{k}, \mathbf{c})$  has a neighborhood  $\mathcal{U}_\sigma \subset P$  such that  $\mathcal{U}_\sigma \cap P(\mathbf{k}, \mathbf{c})$  is the set of all  $\tau \in \mathcal{U}_\sigma$  for which (3.18) holds.*

*Proof.* Since all the operators  $\dot{\mathbf{D}}_\tau^{\theta_i}$  are Fredholm and they depend continuously on  $\tau$ , we can assume  $\dim \ker \dot{\mathbf{D}}_\tau^{\theta_i} \leq \dim \ker \dot{\mathbf{D}}_\sigma^{\theta_i}$  for all  $i = 1, \dots, N$  if  $\tau$  is sufficiently close to  $\sigma$ . Thus (3.18) can only be satisfied if none of these inequalities are strict, which means  $\tau \in P(\mathbf{k}, \mathbf{c})$  since every  $\ell_i$  is positive.  $\square$

Recall from §3.2 that the weighted Sobolev spaces  $W^{k,p,-\delta}(\hat{\varphi}_\tau^* \dot{E}_\tau)$  and  $W^{k-1,p,-\delta}(\hat{\varphi}_\tau^* \dot{F}_\tau)$  are defined in terms of fixed families of trivializations of  $E_\tau$  near  $\Theta_\tau$  and holomorphic cylindrical coordinates which allow us to compute Sobolev norms on the cylindrical ends. Given  $\sigma \in P(\mathbf{k}, \mathbf{c})$ , choose a neighborhood  $\mathcal{U}_\sigma \subset P$  that is diffeomorphic to a ball and small enough to satisfy Lemma 3.24. By assumption the bundles  $E_\tau$  depend smoothly on  $\tau$ , which means there is a well-defined smooth bundle  $\widehat{E} \rightarrow P \times \Sigma$  with  $\widehat{E}_{(\tau,z)} = (E_\tau)_z$ . Choosing a suitable connection on the latter, we can use parallel transport along paths of the form  $(\tau(t), \psi_{\tau(t)}(z)) \in \mathcal{U}_\sigma \times \Sigma$  with  $\tau(t)$  radiating outward from  $\sigma$  to define a smooth family of complex bundle isomorphisms

$$\Psi_\tau : \psi_\sigma^* E_\sigma \rightarrow \psi_\tau^* E_\tau$$

which respect these fixed trivializations near  $\Theta_\tau$  and satisfy  $\Psi_\sigma = \operatorname{Id}$ . These give rise to isomorphisms  $\dot{E}_\sigma \rightarrow \dot{E}_\tau$  covering the diffeomorphisms  $\psi_\tau \circ \psi_\sigma^{-1} : \dot{\Sigma}_\sigma \rightarrow \dot{\Sigma}_\tau$ . Notice that there are also natural real bundle isomorphisms

$$d\psi_\tau : T\Sigma \rightarrow \psi_\tau^* T\Sigma,$$

so that  $d\psi_\tau \circ d\psi_\sigma^{-1}$  gives a family of isomorphisms  $T\dot{\Sigma}_\sigma \rightarrow T\dot{\Sigma}_\tau$  covering  $\dot{\Sigma}_\sigma \xrightarrow{\psi_\tau \circ \psi_\sigma^{-1}} \dot{\Sigma}_\tau$ , and they respect the chosen holomorphic cylindrical coordinates on the ends. These then induce smooth families of isomorphisms of complex bundles over  $\widehat{\Sigma}$ ,

$$\hat{\varphi}_\sigma^* \dot{E}_\sigma \rightarrow \hat{\varphi}_\tau^* \dot{E}_\tau, \quad \hat{\varphi}_\sigma^* \dot{F}_\sigma \rightarrow \hat{\varphi}_\tau^* \dot{F}_\tau$$

which again are the identity for  $\tau = \sigma$  and are also equivariant with respect to the natural  $G$ -action by bundle isomorphisms covering deck transformations of  $\widehat{\Sigma}$ . Acting with these

on sections produces  $\tau$ -parametrized families of  $G$ -equivariant Banach space isomorphisms which we shall also denote by  $\Psi_\tau$ :

$$(3.19) \quad \begin{aligned} W^{k,p,-\delta'}(\hat{\varphi}_\sigma^* \dot{E}_\sigma) &\xrightarrow{\Psi_\tau} W^{k,p,-\delta'}(\hat{\varphi}_\tau^* \dot{E}_\tau), \\ W^{k-1,p,-\delta'}(\hat{\varphi}_\sigma^* \dot{F}_\sigma) &\xrightarrow{\Psi_\tau} W^{k-1,p,-\delta'}(\hat{\varphi}_\tau^* \dot{F}_\tau). \end{aligned}$$

Here  $\Psi_\sigma = \text{Id}$ .

We can now use these isomorphisms to define for  $\tau \in \mathcal{U}_\sigma$  a smooth family of  $G$ -equivariant Fredholm operators with fixed domain and target space,

$$(3.20) \quad \hat{\mathbf{D}}_\tau := \Psi_\tau^{-1} \circ \hat{\varphi}_\tau^* \dot{\mathbf{D}}_\tau \circ \Psi_\tau : W^{k,p,-\delta'}(\hat{\varphi}_\sigma^* \dot{E}_\sigma) \rightarrow W^{k-1,p,-\delta'}(\hat{\varphi}_\sigma^* \dot{F}_\sigma),$$

such that

$$\mathcal{U}_\sigma \cap P(\mathbf{k}, \mathbf{c}) = \left\{ \tau \in \mathcal{U}_\sigma \mid \dim \ker \hat{\mathbf{D}}_\tau = \sum_{i=1}^N \ell_i k_i \right\}.$$

In order to present the latter as the zero-set of a smooth map, let us abbreviate

$$\mathbf{X}_\sigma := W^{k,p,-\delta'}(\hat{\varphi}_\sigma^* \dot{E}_\sigma), \quad \mathbf{Y}_\sigma := W^{k-1,p,-\delta'}(\hat{\varphi}_\sigma^* \dot{F}_\sigma),$$

so (3.20) defines a smooth map

$$\mathcal{U}_\sigma \rightarrow \mathcal{L}_G(\mathbf{X}_\sigma, \mathbf{Y}_\sigma) : \tau \mapsto \hat{\mathbf{D}}_\tau,$$

where  $\mathcal{L}_G(\mathbf{X}_\sigma, \mathbf{Y}_\sigma)$  denotes the Banach space of bounded real-linear maps  $\mathbf{X}_\sigma \rightarrow \mathbf{Y}_\sigma$  that are  $G$ -equivariant. Since  $\hat{\mathbf{D}}_\sigma = \hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma$  is Fredholm, we can choose a splitting

$$\mathbf{X}_\sigma = \mathbf{V}_\sigma \oplus \ker(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma),$$

such that  $\mathbf{V}_\sigma \subset \mathbf{X}_\sigma$  is a closed subspace and  $\hat{\mathbf{D}}_\sigma$  maps  $\mathbf{V}_\sigma$  isomorphically to its image. By Proposition 3.14, we can similarly split

$$\mathbf{Y}_\sigma = \text{im}(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma) \oplus \ker(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*),$$

where  $\ker(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*)$  is equivalently the space of all sections in  $W^{k-1,p,\delta'}(\hat{\varphi}_\sigma^* \dot{F}_\sigma)$  that are  $L^2$ -orthogonal to  $\text{im}(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma)$ . In terms of these splittings,  $\hat{\mathbf{D}}_\tau$  can be written in block form

$$\hat{\mathbf{D}}_\tau = \begin{pmatrix} \mathbf{D}_\tau^{11} & \mathbf{D}_\tau^{12} \\ \mathbf{D}_\tau^{21} & \mathbf{D}_\tau^{22} \end{pmatrix},$$

where after shrinking  $\mathcal{U}_\sigma$  if necessary, we can assume without loss of generality that  $\mathbf{D}_\tau^{11} : \mathbf{V}_\sigma \rightarrow \text{im}(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma)$  is invertible for all  $\tau \in \mathcal{U}_\sigma$ . We can therefore define a map

$$(3.21) \quad \begin{aligned} \mathbf{F}_\sigma : \mathcal{U}_\sigma &\rightarrow \text{Hom}_G(\ker(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma), \ker(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*)) \\ \tau &\mapsto \mathbf{D}_\tau^{22} - \mathbf{D}_\tau^{21}(\mathbf{D}_\tau^{11})^{-1}\mathbf{D}_\tau^{12}. \end{aligned}$$

**Lemma 3.25.** *A parameter  $\tau \in \mathcal{U}_\sigma$  belongs to  $P(\mathbf{k}, \mathbf{c})$  if and only if  $\mathbf{F}_\sigma(\tau) = 0$ .*

*Proof.* Define for each  $\tau \in \mathcal{U}_\sigma$  the Banach space isomorphism

$$\mathbf{T} = \begin{pmatrix} \mathbf{1} & -(\mathbf{D}_\tau^{11})^{-1}\mathbf{D}_\tau^{12} \\ 0 & \mathbf{1} \end{pmatrix} \in \mathcal{L}(\mathbf{V}_\sigma \oplus \ker(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma)) = \mathcal{L}(\mathbf{X}_\sigma).$$

Then  $\widehat{\mathbf{D}}_\tau \mathbf{T} = \begin{pmatrix} \mathbf{D}_\tau^{11} & 0 \\ \mathbf{D}_\tau^{21} & \mathbf{F}_\sigma(\tau) \end{pmatrix}$ , and since  $\mathbf{D}_\tau^{11}$  is invertible,

$$\ker \widehat{\mathbf{D}}_\tau \cong \ker(\widehat{\mathbf{D}}_\tau \mathbf{T}) = \{0\} \oplus \ker \mathbf{F}_\sigma(\tau) \cong \ker \mathbf{F}_\sigma(\tau).$$

The latter can only have the same dimension as  $\ker(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma)$  if  $\mathbf{F}_\sigma(\tau)$  vanishes.  $\square$

Observe that by Lemma 3.22, Corollary 3.23 and Schur's lemma,

$$(3.22) \quad \dim \operatorname{Hom}_G(\ker(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma), \ker(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*)) = \sum_{i=1}^N t_i k_i c_i,$$

where  $t_i = 1, 2, 4$  for representations  $\theta_i$  of real type, complex type and quaternionic type respectively. The lemma implies via the implicit function theorem that a neighborhood of  $\sigma$  in  $P(\mathbf{k}, \mathbf{c})$  is a smooth submanifold with the same codimension that appears in Theorem D whenever we can show that the linearization

$$d\mathbf{F}_\sigma(\sigma) : T_\sigma P \rightarrow \operatorname{Hom}_G(\ker(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma), \ker(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*))$$

is surjective.

We will need a precise formula for this linearization in the following special case. Suppose we have a smooth path

$$\gamma : (-\epsilon, \epsilon) \rightarrow P \quad \text{with } \gamma(0) = \sigma \text{ and } \dot{\gamma}(0) = Y \in T_\sigma P$$

such that for all  $\tau = \gamma(t)$ :

- (1)  $E_\tau = E_\sigma$  (i.e. there is a canonical complex bundle isomorphism);
- (2)  $\psi_\tau = \operatorname{Id}$ ;
- (3)  $j_\tau = j_\sigma$ .

We are then free to choose the bundle isomorphisms  $\Psi_\tau$  and consequently the Banach space isomorphisms (3.19) to be the identity for all  $\tau = \gamma(t)$ , so  $\widehat{\mathbf{D}}_{\gamma(t)} = \hat{\varphi}_\sigma^* \dot{\mathbf{D}}_{\gamma(t)}$ , where  $\mathbf{D}_{\gamma(t)}$  is a smooth family of Cauchy-Riemann operators on the fixed bundle  $E_\sigma \rightarrow \Sigma_\sigma$ . Differentiating this family gives a real-linear bundle map

$$A_Y := \partial_t \mathbf{D}_{\gamma(t)}|_{t=0} \in \Gamma(\operatorname{Hom}_{\mathbb{R}}(E_\sigma, F_\sigma)),$$

and we then find that

$$\mathbf{L}(Y) := d\mathbf{F}_\sigma(\sigma)Y \in \operatorname{Hom}_G(\ker(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma), \ker(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*))$$

takes the form

$$(3.23) \quad \mathbf{L}(Y)\eta = \Pi((\hat{\varphi}_\sigma^* A_Y)\eta),$$

where  $\Pi$  is the projection

$$\mathbf{Y}_\sigma = \operatorname{im}(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma) \oplus \ker(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*) \xrightarrow{\Pi} \ker(\hat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*).$$

The unique continuation result developed in §5 below is geared toward proving that operators such as  $\mathbf{L}$  are surjective.



## 4. INDEX COMPUTATION

The goal of this section is to compute the Fredholm index of the twisted Cauchy-Riemann type operators introduced in §3.4. We will use the notation of §3 but dispense with the parameter  $\tau$  since it is not important for the index computation, hence  $\varphi : (\Sigma', j') \rightarrow (\Sigma, j)$  is a fixed branched cover, and  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$  is a fixed regular presentation. The complex vector bundles  $E$  and  $F$  with their restrictions  $\dot{E}$  and  $\dot{F}$  to the punctured domain  $\dot{\Sigma}$  are assumed to have rank

$$m := \text{rank}_{\mathbb{C}} E \in \mathbb{N},$$

and we assume

$$\theta : G \rightarrow \text{Aut}_{\mathbb{R}}(W)$$

is a real (not necessarily irreducible or faithful) representation of  $G$  with

$$n := \dim W \in \mathbb{N}.$$

The resulting twisted bundles over  $\dot{\Sigma}$  can be written as

$$\dot{E}^{\theta} = \dot{E} \otimes_{\mathbb{R}} W^{\theta}, \quad \dot{F}^{\theta} = \dot{F} \otimes_{\mathbb{R}} W^{\theta},$$

in terms of the flat real vector bundle  $W^{\theta} := (\dot{\Sigma}'' \times W)/G \rightarrow \dot{\Sigma}$ , and any Cauchy-Riemann type operator  $\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}(E)$  then gives rise to the twisted operator

$$\dot{\mathbf{D}}^{\theta} : \Gamma(\dot{E}^{\theta}) \rightarrow \Gamma(\dot{F}^{\theta}),$$

We need a bit more notation in order to state a formula for  $\text{ind}(\dot{\mathbf{D}}^{\theta})$ . Recall that while the deck transformations  $G = \text{Aut}(\pi)$  act on  $\dot{\Sigma}''$  without fixed points, their extensions to biholomorphic self-maps of  $\Sigma''$  may fix some of the punctures, so for each  $w \in \Theta$  and  $\zeta \in \pi^{-1}(w) \subset \Theta'' := \pi^{-1}(\Theta)$ , we can consider the stabilizer subgroup

$$G_{\zeta} := \{g \in G \mid g\zeta = \zeta\},$$

which is necessarily cyclic. Restricting  $\theta$  to  $G_{\zeta}$  then defines a representation  $G_{\zeta} \rightarrow \text{Aut}_{\mathbb{R}}(W)$ , which splits  $W$  into  $G_{\zeta}$ -invariant subspaces  $W = W_{\zeta} \oplus W'_{\zeta}$  such that  $G_{\zeta}$  acts on  $W_{\zeta}$  trivially and on  $W'_{\zeta}$  as a direct sum of nontrivial representations. We define the number

$$n_w := \dim W'_{\zeta} \in \{0, \dots, n\}.$$

As implied by the notation, this depends on  $w \in \Theta$  but not on the choice of preimage  $\zeta \in \pi^{-1}(w)$ : indeed, since  $G$  acts transitively on  $\pi^{-1}(w)$ , any two choices of  $\zeta$  give rise to conjugate subgroups  $G_{\zeta}$ , and using orthonormality of characters, one can compute

$$n_w = n - \dim W_{\zeta} = n - \frac{1}{|G_{\zeta}|} \sum_{g \in G_{\zeta}} \chi_{\theta}(g),$$

an expression which depends only on the conjugacy class of  $G_{\zeta}$ .

**Theorem 4.1.** *Under the assumptions detailed above, the operator*

$$\dot{\mathbf{D}}^{\theta} : W^{k,p,-\delta}(\dot{E}^{\theta}) \rightarrow W^{k-1,p,-\delta}(\dot{F}^{\theta})$$

is Fredholm for any  $k \in \mathbb{N}$ ,  $p \in (1, \infty)$  and negative exponential weights  $-\delta = \{-\delta_w\}_{w \in \Theta}$  satisfying  $0 < \delta_w < 2\pi/|G|$  for all  $w \in \Theta$ . Its index is

$$\text{ind}(\dot{\mathbf{D}}^\theta) = n \cdot \text{ind}(\mathbf{D}) - m \sum_{w \in \Theta} n_w.$$

**Corollary 4.2** (cf. Lemma 2.10). *If  $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$  is the minimal regular presentation and  $\theta$  is faithful, then*

$$\text{ind}(\dot{\mathbf{D}}^\theta) \leq n \cdot \text{ind}(\mathbf{D}) - m|\Theta|,$$

and the inequality is strict unless all branch points of  $\varphi$  have branching order 2.

*Proof.* By Lemma 3.3, the stabilizer subgroups  $G_\zeta$  are nontrivial for all  $\zeta \in \Theta''$ , and the conclusion about branch points of order 2 will hold if and only if all of them are isomorphic to  $\mathbb{Z}_2$ . Now if  $\theta$  is faithful, it follows that all nontrivial elements  $g \in G_\zeta$  for  $\zeta \in \Theta''$  also act nontrivially on  $W$ , hence  $n_w \geq 1$  for all  $w \in \Theta$ . This implies the upper bound, and it is an equality if and only if  $n_w = 1$  for all  $w \in \Theta$ , meaning each  $G_\zeta$  acts on  $W$  as the  $(n-1)$ -fold direct sum of the trivial representation plus a 1-dimensional nontrivial representation, which is required to be faithful. But the only nontrivial faithful real 1-dimensional representation of any finite group is the nontrivial representation of  $\mathbb{Z}_2$ , hence  $G_\zeta \cong \mathbb{Z}_2$ .  $\square$

The remainder of this section is devoted to the proof of Theorem 4.1. It will be convenient first to complexify the representation. We define  $V := W \oplus iW$  and the complex representation

$$\lambda : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$$

such that  $\lambda(g)|_W = \theta(g)$  for all  $g \in G$ . Note that for  $w \in \Theta$  and  $\zeta \in \pi^{-1}(w) \subset \Theta''$ , the trivial representation of  $G_\zeta$  on  $V$  is the complexification of the trivial real representation on  $W$ , so the splitting  $W = W_\zeta \oplus W'_\zeta$  explained above complexifies to a splitting  $V = V_\zeta \oplus V'_\zeta$ , where  $V_\zeta \subset V$  is the largest complex subspace on which  $G_\zeta$  acts trivially, allowing us to write

$$n_w = \dim_{\mathbb{C}} V'_\zeta = n - \dim_{\mathbb{C}} V_\zeta.$$

The complexified representation now gives rise to a complex flat bundle  $V^\lambda := (\dot{\Sigma}'' \times V)/G$ , corresponding twisted bundles

$$(4.1) \quad \dot{E}^\lambda := \dot{E} \otimes_{\mathbb{R}} V^\lambda, \quad \dot{F}^\lambda := \dot{F} \otimes_{\mathbb{R}} V^\lambda,$$

and a twisted Cauchy-Riemann operator

$$\dot{\mathbf{D}}^\lambda : W^{k,p,-\delta}(\dot{E}^\lambda) \rightarrow W^{k-1,p,-\delta}(\dot{F}^\lambda).$$

The following point is important to understand: the tensor products in (4.1) are *real*, thus  $\dot{E}^\lambda$  and  $\dot{F}^\lambda$  each inherit two complex structures  $J$  and  $i$ , where  $J$  comes from the complex structure of  $E$  and  $i$  from that of  $V$ : they commute with each other and are defined by

$$J(\eta \otimes v) := J\eta \otimes v, \quad i(\eta \otimes v) := \eta \otimes iv.$$

In this sense,  $\dot{\mathbf{D}}^\lambda$  can be regarded as the  $i$ -complex-linear extension of  $\dot{\mathbf{D}}^\theta$  to complexifications of the latter's domain and target space—this notion of “complexification” ignores the fact that these spaces already have native complex structures  $J$  and treats them as

real vector spaces, which is appropriate since  $\dot{\mathbf{D}}^\theta$  is not  $J$ -complex linear. We therefore obtain the relation

$$\text{ind}(\dot{\mathbf{D}}^\theta) = \frac{1}{2} \text{ind}(\dot{\mathbf{D}}^\lambda),$$

and we shall compute  $\text{ind}(\dot{\mathbf{D}}^\lambda)$  by regarding  $\dot{\mathbf{D}}^\lambda$  as a real-linear Cauchy-Riemann type operator on the complex vector bundle  $(\dot{E}^\lambda, J)$ . Since  $\text{rank}_{\mathbb{C}} \dot{E}^\lambda = \text{rank}_{\mathbb{C}} E \cdot \dim_{\mathbb{R}} V = 2mn$ , the punctured Riemann-Roch formula from [Sch95] gives

$$(4.2) \quad \text{ind}(\dot{\mathbf{D}}^\lambda) = 2mn \cdot \chi(\dot{\Sigma}) + 2c_1^\Phi(\dot{E}^\lambda, J) + \sum_{w \in \Theta} \mu_{\text{CZ}}^\Phi(\mathbf{A}_w^\lambda - \delta_w),$$

where  $\Phi$  is an arbitrary choice of asymptotic trivialization, and  $\mu_{\text{CZ}}^\Phi(\mathbf{A}_w^\lambda - \delta_w) \in \mathbb{Z}$  are Conley-Zehnder indices that depend on certain asymptotic operators  $\mathbf{A}_w^\lambda$  to be discussed below and the exponential weight  $-\delta_w \in (-2\pi/|G|, 0)$  associated to each puncture  $w \in \Theta$ . The main difficulty of the calculation is in choosing a suitable asymptotic trivialization in which both  $c_1^\Phi(\dot{E}^\lambda, J)$  and  $\mu_{\text{CZ}}^\Phi(\mathbf{A}_w^\lambda - \delta_w)$  can be computed.

Denote

$$d' := \deg(\pi) = |G|,$$

and for each  $w \in \Theta$  and  $\zeta \in \pi^{-1}(w) \subset \Theta''$ , let

$$k_\zeta \in \{1, \dots, d'\}$$

denote the branching order of  $\pi$  at  $\zeta$ , meaning  $\pi$  is a  $k_\zeta$ -to-1 map on a small punctured neighborhood of  $\zeta$ . We can then choose punctured neighborhoods  $\mathcal{U}_w \subset \dot{\Sigma}$  and  $\mathcal{U}_\zeta \subset \dot{\Sigma}''$  of  $w$  and  $\zeta$  respectively, with holomorphic cylindrical coordinates  $(s, t) \in [0, \infty) \times S^1$  on each such that

$$\pi(s, t) = (k_\zeta s, k_\zeta t)$$

in coordinates on  $\mathcal{U}_\zeta$ . In these coordinates, any  $g \in G_\zeta$  necessarily preserves the end  $\mathcal{U}_\zeta$  and takes the form  $g(s, t) = (s, t + j/k_\zeta)$  for some  $j \in \{0, \dots, k_\zeta - 1\}$ . This means that  $G_\zeta$  is a cyclic group of order  $k_\zeta$ , and it has a canonical generator  $g_\zeta \in G_\zeta$  such that

$$g_\zeta(s, t) = (s, t + 1/k_\zeta) \quad \text{on } \mathcal{U}_\zeta.$$

In addition to the cylindrical coordinates, let us choose complex trivializations of  $E$  on each of the corresponding neighborhoods of  $\Theta$ , thus giving an identification

$$(4.3) \quad \dot{E}|_{\mathcal{U}_w} = ([0, \infty) \times S^1) \times E_w$$

for each  $w \in \Theta$ . For any choice  $\zeta \in \pi^{-1}(w) \subset \Theta''$ , this also gives us an identification of  $\dot{E}^\lambda|_{\mathcal{U}_w}$  with

$$(4.4) \quad (([0, \infty) \times S^1) \times (E_w \otimes_{\mathbb{R}} V)) / G_\zeta,$$

where the action of  $G_\zeta = \mathbb{Z}_{k_\zeta}$  on  $([0, \infty) \times S^1) \times (E_w \otimes_{\mathbb{R}} V)$  is determined by

$$g_\zeta \cdot ((s, t), \eta \otimes v) = ((s, t + 1/k_\zeta), \eta \otimes \lambda(g_\zeta)v).$$

This picture can now easily be extended to the “circle compactification” of the punctured surface: let  $\overline{\Sigma}$  and  $\overline{\Sigma}''$  denote the compact surfaces with boundary obtained by replacing each cylindrical end  $[0, \infty) \times S^1$  in  $\dot{\Sigma}$  and  $\dot{\Sigma}''$  respectively by the compact topological manifold  $[0, \infty] \times S^1$ . The connected components of  $\partial \overline{\Sigma}$  and  $\partial \overline{\Sigma}''$  are then in bijective

correspondence with the punctures  $w \in \Theta$  or  $\zeta \in \Theta''$  respectively, and the choice of cylindrical coordinates identifies each of these components with  $S^1$ . We shall denote the boundary components accordingly by  $S_w^1, S_\zeta^1$  for  $w \in \Theta$  or  $\zeta \in \Theta''$ , hence

$$\partial\bar{\Sigma} = \bigsqcup_{w \in \Theta} S_w^1, \quad \partial\bar{\Sigma}'' = \bigsqcup_{\zeta \in \Theta''} S_\zeta^1.$$

The covering map  $\pi : \dot{\Sigma}'' \rightarrow \dot{\Sigma}$  now extends to a continuous covering map

$$\bar{\pi} : \bar{\Sigma}'' \rightarrow \bar{\Sigma}$$

which restricts on the boundary components to

$$\pi_\zeta := \bar{\pi}|_{S_\zeta^1} : S_\zeta^1 \rightarrow S_{\pi(\zeta)}^1 : t \mapsto k_\zeta t,$$

and each  $g \in G$  also extends naturally to a continuous deck transformation  $\bar{g} : \bar{\Sigma}'' \rightarrow \bar{\Sigma}''$  of  $\bar{\pi}$ , such that if  $g(\zeta) = \zeta'$ , then  $\bar{g}$  maps  $S_\zeta^1 \rightarrow S_{\zeta'}^1$  via the canonical diffeomorphism composed with a translation. The identifications (4.3) and (4.4) then yield obvious extensions of  $\dot{E}$  and  $\dot{E}^\lambda$  as topological vector bundles

$$\bar{E} \rightarrow \bar{\Sigma}, \quad \bar{E}^\lambda \rightarrow \bar{\Sigma},$$

and we have

$$\bar{E}^\lambda = (\bar{\pi}^* \bar{E} \otimes_{\mathbb{R}} V) / G.$$

We are now prepared to discuss asymptotic operators. Recall that after choosing a suitable Hermitian inner product on  $\dot{E}$  over the cylindrical ends, any Cauchy-Riemann type operator  $\dot{\mathbf{D}}$  on  $\dot{E} \rightarrow \dot{\Sigma}$  with reasonable asymptotic behavior determines real-linear operators

$$\mathbf{A}_w : \Gamma(\bar{E}|_{S_w^1}) \rightarrow \Gamma(\bar{E}|_{S_w^1}),$$

for each  $w \in \Theta$ , see e.g. [Wen10, §2.1]. These can be regarded as unbounded self-adjoint operators on  $L^2(\bar{E}|_{S_w^1})$  with dense domain  $H^1(\bar{E}|_{S_w^1})$ , and we say  $\mathbf{A}_w$  is **nondegenerate** whenever its kernel is trivial, in which case it determines a **Conley-Zehnder index**

$$\mu_{\text{CZ}}^\Phi(\mathbf{A}_w) \in \mathbb{Z}$$

relative to any choice of complex trivialization  $\Phi$  of  $\bar{E}|_{S_w^1}$ . In the case where  $\dot{\mathbf{D}}$  is the restriction to  $\dot{\Sigma}$  of some operator  $\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}(E)$  on  $\Sigma$ , the operators  $\mathbf{A}_w$  are very simple and were already computed in §3.2: they are each the so-called *trivial* asymptotic operator

$$\mathbf{A}_w = -J\partial_t,$$

where  $\partial_t$  is a well-defined differential operator on  $\bar{E}|_{S_w^1}$  since the fibers are all canonically identified with  $E_w$ . This operator is degenerate, but the introduction of negative exponential weights  $-\delta_w < 0$  identifies  $\dot{\mathbf{D}}$  with another Cauchy-Riemann type operator whose corresponding asymptotic operators are  $\mathbf{A}_w - \delta_w$ , which are nondegenerate for any  $\delta_w > 0$  sufficiently small.

Denote by

$$\mathbf{A}_w^\lambda : \Gamma(\bar{E}^\lambda|_{S_w^1}) \rightarrow \Gamma(\bar{E}^\lambda|_{S_w^1})$$

the asymptotic operators associated to  $\dot{\mathbf{D}}^\lambda$  for each  $w \in \Theta$ . These are easiest to understand by considering the pulled back Cauchy-Riemann operator

$$\pi^* \dot{\mathbf{D}}^\lambda : W^{1,p,-\pi^* \delta}(\pi^* \dot{E}^\lambda) \rightarrow L^{p,-\pi^* \delta}(\pi^* \dot{F}^\lambda),$$

whose asymptotic operators we will denote by

$$\pi^* \mathbf{A}_\zeta^\lambda : \Gamma((\bar{\pi}^* \bar{E}^\lambda)|_{S_\zeta^1}) \rightarrow \Gamma((\bar{\pi}^* \bar{E}^\lambda)|_{S_\zeta^1})$$

for  $\zeta \in \Theta''$ . The relation  $(\pi^* \dot{\mathbf{D}}^\lambda)(\eta \circ \pi) = \pi^*(\dot{\mathbf{D}}^\lambda \eta)$  for sections  $\eta \in \Gamma(\dot{E}^\lambda)$  gives rise to the following relation between asymptotic operators:

$$(4.5) \quad (\pi^* \mathbf{A}_\zeta^\lambda)(f \circ \pi_\zeta) = k_\zeta \cdot (\mathbf{A}_w^\lambda f) \circ \pi_\zeta \quad \text{for } f \in \Gamma(\bar{E}^\lambda|_{S_w^1}) \text{ and } \zeta \in \pi^{-1}(w).$$

For the following discussion, fix  $w \in \Theta$  and  $\zeta \in \pi^{-1}(w)$ . The definition of  $\dot{\mathbf{D}}^\lambda$  implies that  $\pi^* \dot{\mathbf{D}}^\lambda$  acts on sections  $\eta \otimes v \in \Gamma(\pi^* \dot{E} \otimes_{\mathbb{R}} V)$  such that  $(\pi^* \dot{\mathbf{D}}^\lambda)(\eta \otimes v) = [(\pi^* \dot{\mathbf{D}}) \eta] \otimes v$  whenever  $v : \dot{S}'' \rightarrow V$  is constant. From this, one deduces that for any section  $f \otimes v \in \Gamma(\bar{\pi}^* \bar{E} \otimes_{\mathbb{R}} V|_{S_\zeta^1})$  where  $f$  is an arbitrary smooth map  $S_\zeta^1 \rightarrow E_w$  and  $v : S_\zeta^1 \rightarrow V$  is constant, we have

$$(4.6) \quad \pi^* \mathbf{A}_\zeta^\lambda(f \otimes v) = -(J \partial_t f) \otimes v.$$

Now to write down a formula for  $\mathbf{A}_w^\lambda$ , we can use the natural identification of  $\Gamma(\bar{E}^\lambda|_{S_w^1})$  with the space of  $G_\zeta$ -equivariant loops in  $E_w \otimes_{\mathbb{R}} V$ ,

$$\Gamma(\bar{E}^\lambda|_{S_w^1}) = \left\{ F \in C^\infty(S^1, E_w \otimes_{\mathbb{R}} V) \mid F(t + 1/k_\zeta) = g_\zeta \cdot F(t) \text{ for all } t \in S^1 \right\}.$$

Acting on  $G_\zeta$ -equivariant loops  $F$ , (4.5) and (4.6) imply

$$(4.7) \quad \mathbf{A}_w^\lambda F = -\frac{1}{k_\zeta} J \partial_t F,$$

where it is understood that  $J \partial_t$  acts on the tensor product by taking  $F = f \otimes v$  to  $(J \partial_t f) \otimes v$  whenever  $v$  is locally constant.

In order to choose a suitable trivialization  $\Phi$  and compute  $\mu_{\text{CZ}}^\Phi(\mathbf{A}_w^\lambda - \delta_w)$ , we shall split  $\mathbf{A}_w^\lambda$  into a direct sum of operators on  $J$ -complex line bundles. First, we observe that  $\bar{E}|_{S_w^1} = S_w^1 \times E_w$  is already canonically trivial, so any complex basis of  $E_w$  gives a splitting of  $\mathbf{A}_w^\lambda$  over an  $m$ -fold direct sum of isomorphic  $J$ -complex bundles of rank  $2n$ ,

$$\bar{E}^\lambda|_{S_w^1} = (L^\lambda)^{\oplus m},$$

where

$$L^\lambda = S^1 \times (\mathbb{C} \otimes_{\mathbb{R}} V) / G_\zeta$$

and the generator of  $G_\zeta = \mathbb{Z}_{k_\zeta}$  acts by  $g_\zeta \cdot (t, f \otimes v) = (t + 1/k_\zeta, f \otimes \lambda(g_\zeta)v)$ . Note that  $L^\lambda$  carries two commuting complex structures,  $J$  and  $i$ , which act on the first and second factor of the tensor product respectively. Further:  $V$  admits a complex basis  $(v_1, \dots, v_n)$  consisting of eigenvectors of  $\lambda(g_\zeta)$ , and we can then define integers  $p_j \in \{0, \dots, k_\zeta - 1\}$  for  $j = 1, \dots, n$  by

$$\lambda(g_\zeta)v_j = e^{2\pi i p_j / k_\zeta} v_j.$$

Here we can identify  $V'_\zeta \subset V$  as the subspace spanned by all  $v_j$  such that  $p_j > 0$ . Identifying  $V$  with  $\mathbb{C}^n$  via this eigenbasis yields a splitting

$$L^\lambda = L_1^\lambda \oplus \dots \oplus L_n^\lambda,$$

where for  $j = 1, \dots, n$ ,

$$L_j^\lambda := S^1 \times (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) / \mathbb{Z}_{k_\zeta},$$

with the generator  $1 \in \mathbb{Z}_{k_\zeta}$  acting by  $1 \cdot (t, f \otimes v) = (t + 1/k_\zeta, f \otimes e^{2\pi i p_j / k_\zeta} v)$ . This bundle again carries the two commuting complex structures  $J$  and  $i$  acting on the first and second factors of the tensor product respectively; it has complex rank 2 with respect to either one. Finally, since  $J$  acts  $i$ -complex-linearly on  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ , we can find eigenvectors  $f_\pm \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$  such that  $Jf_\pm = \pm i f_\pm$ , so the splitting  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}f_+ \oplus \mathbb{C}f_-$  gives a splitting of  $J$ - and  $i$ -complex vector bundles

$$L_j^\lambda = L_{j,+}^\lambda \oplus L_{j,-}^\lambda,$$

with

$$(4.8) \quad L_{j,\pm}^\lambda = (S^1 \times \mathbb{C}) / \mathbb{Z}_{k_\zeta},$$

where the generator  $1 \in \mathbb{Z}_{k_\zeta}$  acts by  $1 \cdot (t, f) = (t + 1/k_\zeta, e^{2\pi i p_j / k_\zeta} f)$ . Both  $L_{j,+}^\lambda$  and  $L_{j,-}^\lambda$  are complex line bundles over  $S^1$ , carrying two complex structures  $J$  and  $i$ , which satisfy  $J = i$  on  $L_{j,+}^\lambda$  but  $J = -i$  on  $L_{j,-}^\lambda$ . This splitting of bundles gives a splitting of  $\mathbf{A}_w^\lambda$  in the form

$$(4.9) \quad \mathbf{A}_w^\lambda = \left( \bigoplus_{j=1}^n (\mathbf{A}_{j,+}^\lambda \oplus \mathbf{A}_{j,-}^\lambda) \right)^{\oplus m},$$

where for  $j = 1, \dots, n$ ,  $\mathbf{A}_{j,\pm}^\lambda$  acts on

$$\Gamma(L_{j,\pm}^\lambda) = \left\{ f \in C^\infty(S^1, \mathbb{C}) \mid f(t + 1/k_\zeta) = e^{2\pi i p_j / k_\zeta} f(t) \text{ for all } t \in S^1 \right\}$$

by

$$\mathbf{A}_{j,\pm}^\lambda f = \mp \frac{1}{k_\zeta} i \partial_t f.$$

Since  $L_{j,\pm}^\lambda$  are complex line bundles,  $\mu_{\text{CZ}}^\Phi(\mathbf{A}_{j,\pm}^\lambda - \delta_w)$  can be computed in terms of winding numbers of eigenfunctions of  $\mathbf{A}_{j,\pm}^\lambda$ , using the relation proved in [HWZ95, Theorem 3.10]. In particular, if (as will turn out to be true in our case) all eigenspaces of  $\mathbf{A}_{j,\pm}^\lambda$  have real dimension 2, then

$$(4.10) \quad \mu_{\text{CZ}}^\Phi(\mathbf{A}_{j,\pm}^\lambda - \delta_w) = 2 \text{wind}^\Phi(f_{j,\pm}) + 1,$$

where  $f_{j,\pm} \in \Gamma(L_{j,\pm}^\lambda)$  is any nontrivial eigenfunction of  $\mathbf{A}_{j,\pm}^\lambda - \delta_w$  with the largest possible negative eigenvalue. A  $\mathbb{Z}_{k_\zeta}$ -equivariant function  $f : S^1 \rightarrow \mathbb{C}$  satisfies  $\mathbf{A}_{j,\pm}^\lambda f = \lambda f$  if and only if it is a complex multiple of

$$(4.11) \quad f_\lambda(t) := e^{\pm i k_\zeta \lambda t}, \quad \lambda \mp \frac{2\pi p_j}{k_\zeta} \in 2\pi\mathbb{Z}.$$

Observe that since  $0 < \delta_w < 2\pi/d' \leq 2\pi/k_\zeta$ , every eigenvalue  $\lambda$  thus satisfies  $\lambda - \delta_w \neq 0$ ; this proves that the perturbed asymptotic operators  $\mathbf{A}_{j,\pm}^\lambda$  are all nondegenerate and thus establishes the Fredholm property for  $\dot{\mathbf{D}}^\lambda$ . Now to apply (4.10), we need to find the unique eigenvalue  $\lambda = 2\pi(\ell \pm p_j/k_\zeta)$  for  $\ell \in \mathbb{Z}$  such that

$$2\pi \left( \ell \pm \frac{p_j}{k_\zeta} \right) - \delta_w < 0 < 2\pi \left[ (\ell + 1) \pm \frac{p_j}{k_\zeta} \right] - \delta_w.$$

Since  $0 < \delta_w < 2\pi/d'$ , this condition is equivalent to

$$\ell \leq \mp \frac{p_j}{k_\zeta} < \ell + 1,$$

so choosing the appropriate  $\ell \in \mathbb{Z}$  and plugging in (4.11) leads to the formulas

$$(4.12) \quad \begin{aligned} f_{j,+}(t) &:= \begin{cases} 1 & \text{if } p_j = 0, \\ e^{-2\pi i(k_\zeta - p_j)t} & \text{if } p_j > 0, \end{cases} \\ f_{j,-}(t) &:= e^{2\pi i p_j t}. \end{aligned}$$

Let  $\Phi_j^\pm$  for  $j = 1, \dots, n$  denote a choice of  $J$ -complex trivializations of  $L_{j,\pm}^\lambda$  such that

$$\text{wind}^{\Phi_j^+}(f_{j,+}) = \text{wind}^{\Phi_j^-}(f_{j,-}) = 0, \quad j = 1, \dots, n,$$

and denote by  $\Phi_w$  the resulting  $J$ -complex trivialization of

$$(4.13) \quad \overline{E}^\lambda|_{S_w^1} = \left( \bigoplus_{j=1}^n (L_{j,+}^\lambda \oplus L_{j,-}^\lambda) \right)^{\oplus m}.$$

By (4.10), we now have

$$\mu_{\text{CZ}}^{\Phi_j^+}(\mathbf{A}_{j,+}^\lambda - \delta_w) = \mu_{\text{CZ}}^{\Phi_j^-}(\mathbf{A}_{j,-}^\lambda - \delta_w) = 1,$$

and thus by (4.9),

$$\mu_{\text{CZ}}^{\Phi_w}(\mathbf{A}_w^\lambda - \delta_w) = 2mn.$$

Note that, a priori, this construction of  $\Phi_w$  depends on an arbitrary choice  $\zeta \in \pi^{-1}(w)$ , but the fact that  $\mu_{\text{CZ}}^{\Phi_w}(\mathbf{A}_w^\lambda - \delta_w)$  turns out to be independent of this choice tells us that  $\Phi_w$  is uniquely determined up to homotopy. Performing this construction for all punctures  $w \in \Theta$ , we will denote the resulting asymptotic trivialization of  $\dot{E}^\lambda$  simply by  $\Phi$ .

It remains to compute  $c_1^\Phi(\dot{E}^\lambda, J)$ . Consider the pullback  $\pi^* \dot{E}^\lambda = \pi^* \dot{E} \otimes_{\mathbb{R}} V$ . The first factor in this tensor product has a canonical asymptotic trivialization, which we shall denote by  $\pi^* \Psi_0$ , as it is the pullback of an asymptotic trivialization  $\Psi_0$  for  $\dot{E}$ , satisfying  $c_1^{\Psi_0}(\dot{E}) = c_1(E)$ . Moreover, the second factor is globally trivial, thus  $\pi^* \dot{E}^\lambda$  carries a canonical asymptotic trivialization, denoted by  $\Psi$ , such that

$$c_1^\Psi(\pi^* \dot{E}^\lambda) = \dim_{\mathbb{R}} V \cdot c_1^{\pi^* \Psi_0}(\pi^* \dot{E}) = 2n \cdot \deg(\pi) \cdot c_1^{\Psi_0}(\dot{E}) = 2nd' \cdot c_1(E).$$



If  $\pi^*\Phi$  denotes the pullback of  $\Phi$  to an asymptotic trivialization of  $\pi^*\dot{E}^\lambda$ , we then have

$$(4.14) \quad \begin{aligned} c_1^\Phi(\dot{E}^\lambda) &= \frac{1}{d'} c_1^{\pi^*\Phi}(\pi^*\dot{E}^\lambda) = \frac{1}{d'} \left[ c_1^\Psi(\pi^*\dot{E}^\lambda) + \deg^\Psi(\pi^*\Phi) \right] \\ &= 2n \cdot c_1(E) + \frac{1}{d'} \deg^\Psi(\pi^*\Phi), \end{aligned}$$

where  $\deg^\Psi(\pi^*\Phi) \in \mathbb{Z}$  denotes the sum over all punctures  $\zeta \in \Theta''$  of the degrees of the transition maps  $S^1 \rightarrow \mathrm{GL}(2mn, \mathbb{C})$  that change  $\Psi$  to  $\pi^*\Phi$ . We can compute the latter for each  $w \in \Theta$  and  $\zeta \in \pi^{-1}(w) \subset \Theta''$  as a sum of winding numbers over a line bundle decomposition analogous to (4.13), namely

$$\bar{\pi}^*\bar{E}^\lambda|_{S_\zeta^1} = \pi_\zeta^* \left( \bar{E}^\lambda|_{S_w^1} \right) = \left( \bigoplus_{j=1}^n \left( \pi_\zeta^* L_{j,+}^\lambda \oplus \pi_\zeta^* L_{j,-}^\lambda \right) \right)^{\oplus m},$$

where pulling back (4.8) via the projection  $\pi_\zeta : S^1 \rightarrow S^1/\mathbb{Z}_{k_\zeta}$  gives the trivial line bundle

$$\pi_\zeta^* L_{j,\pm}^\lambda = S^1 \times \mathbb{C},$$

with the pulled back trivialization  $\pi_\zeta^* \Phi_j^\pm$  such that the special eigenfunctions  $f_{j,\pm}$  in (4.12) have zero winding as  $t$  traverses  $S^1$ . The restriction  $\Psi_\zeta$  of  $\Psi$  to  $\bar{\pi}^*\bar{E}^\lambda|_{S_\zeta^1}$  is now the direct sum of the standard trivializations on each of the factors  $\pi_\zeta^* L_{j,\pm}^\lambda$ , thus

$$(4.15) \quad \deg^{\Psi_\zeta}(\pi_\zeta^* \Phi_w) = m \sum_{j=1}^n [\mathrm{wind}_{S^1}(f_{j,+}) + \mathrm{wind}_{S^1}(f_{j,-})].$$

There is an important subtlety here: recall that  $J = \pm i$  on  $L_{j,\pm}^\lambda$ , hence the orientation induced by  $J$  on  $L_{j,-}^\lambda$  is the *opposite* of the obvious one, and the sign of  $\mathrm{wind}_{S^1}(f_{j,-})$  must be reversed accordingly, giving

$$\begin{aligned} \mathrm{wind}_{S^1}(f_{j,+}) &= \begin{cases} 0 & \text{if } p_j = 0, \\ p_j - k_\zeta & \text{if } p_j > 0, \end{cases} \\ \mathrm{wind}_{S^1}(f_{j,-}) &= -p_j. \end{aligned}$$

Plugging this into (4.15), we have

$$\deg^{\Psi_\zeta}(\pi_\zeta^* \Phi_w) = m \sum_{j \in \{1, \dots, n\}, p_j \neq 0} (-k_\zeta) = -mk_\zeta \dim_{\mathbb{C}} V'_\zeta.$$

Summing over all  $\zeta \in \Theta''$  and plugging into (4.14) then gives

$$c_1^\Phi(\dot{E}^\lambda) = 2n \cdot c_1(E) - \frac{m}{d'} \sum_{\zeta \in \Theta''} k_\zeta \dim_{\mathbb{C}} V'_\zeta = 2n \cdot c_1(E) - m \sum_{w \in \Theta} n_w,$$

where we've used the fact that  $\dim_{\mathbb{C}} V'_\zeta = n_w$  is independent of  $\zeta \in \pi^{-1}(w)$  for each  $w \in \Theta$ , and  $\sum_{\zeta \in \pi^{-1}(w)} k_\zeta = d'$ .

Finally, we combine the above expression with the Conley-Zehnder index and plug into (4.2) to obtain

$$\text{ind}(\dot{\mathbf{D}}^\lambda) = 2mn \cdot \chi(\dot{\Sigma}) + 4n \cdot c_1(E) - 2m \sum_{w \in \Theta} n_w,$$

and thus

$$\text{ind}(\dot{\mathbf{D}}^\theta) = n [m\chi(\Sigma) + 2c_1(E)] - m \sum_{w \in \Theta} n_w.$$

The expression in brackets is  $\text{ind } \mathbf{D}$ , so this completes the proof of Theorem 4.1.

## 5. UNIQUE CONTINUATION FOR TENSOR PRODUCTS

Standard proofs of transversality results via the Sard-Smale theorem (cf. [FHS95, MS04]) typically require some kind of unique continuation lemma, which for  $J$ -holomorphic curves usually means the similarity principle. A less standard result of this type is also needed for the proof of Theorem D, and is the topic of the present section.

The following is a local result, so we do not need to make any global assumptions about the Riemann surface  $\Sigma$  or vector bundles  $E$  and  $F$ . In analogy with Cauchy-Riemann type operators on a vector bundle  $E \rightarrow \Sigma$  over a Riemann surface, we shall refer to a real-linear first-order partial differential operator  $\mathbf{D} : \Gamma(E) \rightarrow \Omega^{1,0}(\Sigma, E)$  as an **anti-Cauchy-Riemann type operator** if it satisfies the Leibniz rule

$$\mathbf{D}(f\eta) = (\partial f)\eta + f\mathbf{D}\eta \quad \text{for all } f \in C^\infty(\Sigma, \mathbb{R}), \eta \in \Gamma(E),$$

where  $\partial f := df - i df \circ j \in \Omega^{1,0}(\Sigma, E)$ . Two classes of examples immediately come to mind: first if  $\mathbf{D}$  is a Cauchy-Riemann type operator on  $E$  and we denote by  $\overline{E}$  the same real bundle with its complex structure modified by a sign change, then  $\mathbf{D}$  defines an anti-Cauchy-Riemann type operator on  $\overline{E}$ . More importantly for our purposes, the formal adjoint  $\mathbf{D}^*$  can be regarded as an anti-Cauchy-Riemann operator on  $F := \overline{\text{Hom}_{\mathbb{C}}(T\Sigma, E)}$  after using the relevant bundle metrics to identify  $\Gamma(E)$  with  $\Omega^{1,0}(\Sigma, F)$ .

We say that a smooth section of a vector bundle  $E \rightarrow \Sigma$  **vanishes to infinite order** at a point  $z \in \Sigma$  if in some choice of local trivialization near  $z$ , its derivatives of all orders vanish at  $z$ . A section that does not vanish to all orders at any point has the property that it is nonzero on an open and dense subset of  $\Sigma$ . The standard unique continuation result for Cauchy-Riemann type equations states that nontrivial solutions to such equations do not vanish to infinite order at any point. Note that by complex conjugation, the same is true of solutions to anti-Cauchy-Riemann type equations.

**Proposition 5.1.** *Suppose  $E$  and  $F$  are smooth finite-rank complex vector bundles over a Riemann surface  $\Sigma$ , with a Cauchy-Riemann type operator  $\mathbf{D} : \Gamma(E) \rightarrow \Omega^{0,1}(\Sigma, E)$  and an anti-Cauchy-Riemann type operator  $\mathbf{D}' : \Gamma(F) \rightarrow \Omega^{1,0}(\Sigma, F)$ . Then for any sets of complex-linearly independent sections*

$$\eta_1, \dots, \eta_p \in \ker \mathbf{D} \quad \text{and} \quad \xi_1, \dots, \xi_q \in \ker \mathbf{D}',$$

*no linear combination*

$$\sum_{i,j} c^{ij} \xi_i \otimes \eta_j \in \Gamma(F \otimes_{\mathbb{C}} E)$$

with coefficients  $c^{ij} \in \mathbb{C}$  satisfying  $c^{ij} \neq 0$  for some  $i, j$  vanishes to infinite order at any point.

Recall from Lemma 3.11 that every Cauchy-Riemann type operator can be perturbed to one whose kernel and cokernel are *totally real*, meaning they have real bases that are also complex-linearly independent.

**Corollary 5.2.** *Under the assumptions of Proposition 5.1, suppose  $K \subset \ker \mathbf{D}$  and  $C \subset \ker \mathbf{D}'$  are finite-dimensional totally real subspaces of  $\Gamma(E)$  and  $\Gamma(F)$  respectively. Then the natural map*

$$C \otimes_{\mathbb{R}} K \xrightarrow{\iota} \Gamma(F \otimes_{\mathbb{R}} E)$$

*defined by  $\iota(\xi \otimes \eta)(z) = \xi(z) \otimes \eta(z)$  is injective, and the nontrivial sections in its image do not vanish to infinite order at any point.*

*Proof.* Denote the complex structures on  $E$  and  $F$  by  $J_E$  and  $J_F$  respectively. The proposition implies that the natural map  $C \otimes_{\mathbb{R}} K \rightarrow \Gamma(F \otimes_{\mathbb{C}} E)$  has the stated property, but this map factors through  $\iota$  and the bundle map

$$F \otimes_{\mathbb{R}} E \rightarrow F \otimes_{\mathbb{C}} E = (F \otimes_{\mathbb{R}} E) / \sim$$

defined via the equivalence relation  $\xi \otimes J_E \eta \sim J_F \xi \otimes \eta$ .  $\square$

Before proving Proposition 5.1, let us sketch why it is true in the simplest case, where  $\mathbf{D}$  and  $\mathbf{D}'$  are both complex linear. We can then write these as  $\bar{\partial}$  and  $\partial$  respectively in some local trivialization, so we are left with proving that for any complex-linearly independent sets of holomorphic functions  $\eta_i : \mathbb{D} \rightarrow \mathbb{C}^m$  and antiholomorphic functions  $\xi_j : \mathbb{D} \rightarrow \mathbb{C}^n$ , nontrivial linear combinations  $\sum_{i,j} c^{ij} \xi_i \otimes \eta_j : \mathbb{D} \rightarrow \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^m$  cannot vanish to infinite order at 0. The key is to look at the Taylor series for  $\eta_i$  and  $\xi_j$ : write

$$\eta_i(z) = \sum_{k=0}^{\infty} z^k a_{i,k}, \quad \xi_j(z) = \sum_{k=0}^{\infty} \bar{z}^k b_{j,k}$$

for  $a_{i,k} \in \mathbb{C}^m$  and  $b_{j,k} \in \mathbb{C}^n$ , and note that linear independence means there do not exist any constants  $c_i \in \mathbb{C}$  such that  $\sum_i c_i a_{i,k} = 0$  for all  $k$  or  $\sum_j c_j b_{j,\ell} = 0$  for all  $\ell$  except the trivial case with all  $c_i = 0$ . We then have

$$\sum_{i,j} c^{ij} \xi_i \otimes \eta_j = \sum_{k,\ell=0}^{\infty} z^k \bar{z}^{\ell} \sum_{i,j} c^{ij} (b_{i,\ell} \otimes a_{j,k}),$$

which vanishes to infinite order at 0 if and only if  $\sum_{i,j} c^{ij} (b_{i,\ell} \otimes a_{j,k}) = \sum_i b_{i,\ell} \otimes \left( \sum_j c^{ij} a_{j,k} \right) = 0$  for every  $k, \ell$ . The linear independence of  $\xi_i$  then implies  $\sum_j c^{ij} a_{j,k} = 0$  for all  $i$  and  $k$ , so that linear independence of  $\eta_j$  implies in turn that  $c^{ij} = 0$  for all  $i, j$ .

The above argument does not work in the general case since the Taylor series of  $\eta_i$  and  $\xi_j$  at 0 cannot be assumed to be purely holomorphic and antiholomorphic respectively. We shall therefore modify the argument to use only information about the *first nontrivial term* in such Taylor expansions. Note that in suitable choices of local coordinates and

trivializations, the operators  $\mathbf{D}$  and  $\mathbf{D}'$  in the lemma can always be identified with

$$(5.1) \quad \begin{aligned} \mathbf{D} &= \bar{\partial} + A : C^\infty(\mathbb{D}, \mathbb{C}^m) \rightarrow C^\infty(\mathbb{D}, \mathbb{C}^m), \\ \mathbf{D}' &= \partial + B : C^\infty(\mathbb{D}, \mathbb{C}^n) \rightarrow C^\infty(\mathbb{D}, \mathbb{C}^n), \end{aligned}$$

where

$$\bar{\partial} := \frac{\partial}{\partial \bar{z}}, \quad \partial := \frac{\partial}{\partial z},$$

and  $A, B$  are smooth functions on the disk  $\mathbb{D} \subset \mathbb{C}$  valued in  $\text{End}_{\mathbb{R}}(\mathbb{C}^m)$  and  $\text{End}_{\mathbb{R}}(\mathbb{C}^n)$  respectively.

**Lemma 5.3.** *If  $A$  and  $B$  are as described above, then for any nontrivial solutions  $\eta : \mathbb{D} \rightarrow \mathbb{C}^m$  and  $\xi : \mathbb{D} \rightarrow \mathbb{C}^n$  to the equations  $(\bar{\partial} + A)\eta = 0$  and  $(\partial + B)\xi = 0$ , the first nontrivial terms in the Taylor series for  $\eta$  and  $\xi$  at 0 are holomorphic and antiholomorphic respectively.*

*Proof.* Suppose  $\bar{\partial}\eta(z) = -A(z)\eta(z)$  and  $\partial^\alpha \eta(0) = 0$  for all multiindices  $\alpha$  of order  $|\alpha| \leq k$  but not for all  $|\alpha| = k+1$ . Then if  $|\alpha| = k$ , we have

$$\bar{\partial}\partial^\alpha \eta(0) = \partial^\alpha \bar{\partial}\eta(0) = -\partial^\alpha (A\eta)|_{z=0} = 0,$$

leaving  $\partial^{k+1}\eta(0)$  as the only nonzero derivative of order  $k+1$ . A similar argument using the equation  $\partial\xi(z) = -B(z)\xi(z)$  proves that if  $\xi$  has vanishing derivatives at 0 up to order  $k$  but not  $k+1$ , then its only nonzero derivative of order  $k+1$  is  $\bar{\partial}^{k+1}\xi(0)$ .  $\square$

*Proof of Proposition 5.1.* Choosing suitable local coordinates and trivializations, we shall assume the  $\eta_i$  and  $\xi_j$  are functions  $\mathbb{D} \rightarrow \mathbb{C}^m$  and  $\mathbb{D} \rightarrow \mathbb{C}^n$  respectively, with  $\mathbf{D}$  and  $\mathbf{D}'$  written as in (5.1). Suppose there exist constants  $c^{ij} \in \mathbb{C}$  such that  $\sum_{i,j} c^{ij} \xi_i \otimes \eta_j$  vanishes to infinite order at 0. We then claim that

$$(5.2) \quad \sum_{i,j} c^{ij} \partial^\alpha \xi_i(0) \otimes \partial^\beta \eta_j(0) = 0$$

for all multiindices  $\alpha$  and  $\beta$ . We shall prove this by induction on the total order  $|\alpha| + |\beta|$  of the multiindices. The claim clearly holds for  $|\alpha| = |\beta| = 0$ , so for a given  $k \in \mathbb{N}$ , assume it holds for all pairs of multiindices satisfying  $|\alpha| + |\beta| \leq k-1$ . Denote the individual complex components of  $\eta_i : \mathbb{D} \rightarrow \mathbb{C}^m$  and  $\xi_j : \mathbb{D} \rightarrow \mathbb{C}^n$  by  $\eta_i^\mu$  and  $\xi_j^\nu$  for  $\mu = 1, \dots, m$ ,  $\nu = 1, \dots, n$ . Then (5.2) is equivalent to the relation

$$\sum_{i,j} c^{ij} \cdot \partial^\alpha \xi_i^\nu(0) \cdot \partial^\beta \eta_j^\mu(0) = 0$$

for all  $\mu, \nu$ . If this is true whenever  $|\alpha| + |\beta| \leq k-1$ , it implies

$$(5.3) \quad \partial^\beta \eta_\alpha^\nu(0) = 0 \quad \text{where} \quad \eta_\alpha^\nu := \sum_j \left( \sum_i c^{ij} \partial^\alpha \xi_i^\nu(0) \right) \eta_j,$$

and

$$(5.4) \quad \partial^\alpha \xi_\beta^\mu(0) = 0 \quad \text{where} \quad \xi_\beta^\mu := \sum_i \left( \sum_j c^{ij} \partial^\beta \eta_j^\mu(0) \right) \xi_i,$$

again for all  $\mu, \nu$  whenever  $|\alpha| + |\beta| \leq k - 1$ . Observe that  $\eta_\alpha^\nu$  and  $\xi_\beta^\mu$  are each linear combinations of functions in  $\ker \mathbf{D}$  and  $\ker \mathbf{D}'$  respectively and thus also belong to these kernels. Fixing a multiindex  $\alpha$  with  $|\alpha| = \ell \leq k - 1$ , (5.3) implies that  $\eta_\alpha^\nu$  vanishes up to order  $k - 1 - \ell$  at 0, so Lemma 5.3 implies that  $\partial^\beta \eta_\alpha^\nu(0) = 0$  also holds for all  $|\beta| \leq k - \ell$  with the possible exception of the holomorphic Taylor coefficient  $\partial^{k-\ell} \eta_\alpha^\nu(0)$ . Similarly, if  $\beta$  is a fixed multiindex with  $|\beta| = \ell \leq k - 1$ , then (5.4) implies that  $\xi_\beta^\mu$  vanishes up to order  $k - 1 - \ell$  at 0, and Lemma 5.3 extends this to order  $k - \ell$  with the possible exception of the antiholomorphic coefficient  $\bar{\partial}^{k-\ell} \xi_\beta^\mu(0)$ . This proves all cases of (5.2) with  $|\alpha| + |\beta| = k$  except those for which  $\partial^\alpha$  is a power of  $\bar{\partial}$  and  $\partial^\beta$  is a power of  $\partial$ . Now if  $\gamma$  is any multiindex with  $|\gamma| = k$ , the assumption that  $\sum_{i,j} c^{ij} \xi_i \otimes \eta_j$  vanishes to infinite order at 0 implies

$$0 = \partial^\gamma \sum_{i,j} c^{ij} \xi_i \otimes \eta_j \Big|_{z=0} = \sum_{\alpha+\beta=\gamma} n_{\alpha,\beta} c^{ij} \partial^\alpha \xi_i(0) \otimes \partial^\beta \eta_j(0)$$

for suitable combinatorial constants  $n_{\alpha,\beta} \in \mathbb{N}$ , and this expansion contains exactly one term that has not already been shown to vanish. This implies all remaining cases of (5.2) with  $|\alpha| + |\beta| = k$  and thus completes the inductive argument.

The claim implies that the functions  $\eta_\alpha^\nu$  and  $\xi_\beta^\mu$  defined above all vanish to infinite order at 0, and since they satisfy  $\mathbf{D}\eta_\alpha^\nu = 0$  and  $\mathbf{D}'\xi_\beta^\mu = 0$  respectively, the usual unique continuation results imply that they all vanish identically. Since the sets  $\eta_j$  and  $\xi_i$  are linearly independent, (5.3) then implies

$$\sum_i c^{ij} \partial^\alpha \xi_i(0) = \partial^\alpha \sum_i c^{ij} \xi_i \Big|_{z=0} = 0 \quad \text{for all } j \text{ and } \alpha,$$

while (5.4) implies

$$\sum_j c^{ij} \partial^\beta \eta_j(0) = \partial^\beta \sum_j c^{ij} \eta_j \Big|_{z=0} = 0 \quad \text{for all } i \text{ and } \beta.$$

Since  $\sum_i c^{ij} \xi_i$  and  $\sum_j c^{ij} \eta_j$  each satisfy Cauchy-Riemann or anti-Cauchy-Riemann type equations, we can again appeal to unique continuation and conclude that both are identically zero, which contradicts the linear independence assumption unless all  $c^{ij} = 0$ .  $\square$

We now give an application of Corollary 5.2 which will be crucial for the proof of Theorem D. The setting is as in §3.2: assume  $(\Sigma, j)$  is a closed Riemann surface with a finite set of points  $\Theta \subset \Sigma$ , denote by  $\dot{\Sigma}$  the punctured surface  $\Sigma \setminus \Theta$ , assume  $E \rightarrow \Sigma$  is a complex vector bundle of rank  $m \in \mathbb{N}$ ,  $F := \overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, E)$ , and

$$\dot{E}, \dot{F} \rightarrow \dot{\Sigma}$$

denote the restrictions of  $E$  and  $F$  to the punctured surface. Fixing holomorphic cylindrical coordinates near the punctures and a Cauchy-Riemann type operator  $\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}(E)$ , its restriction to the punctured domain defines a bounded linear map of weighted Sobolev spaces

$$\dot{\mathbf{D}} : W^{k,p,-\delta}(\dot{E}) \rightarrow W^{k-1,p,-\delta}(\dot{F})$$

for all  $k \in \mathbb{N}$ ,  $p \in (1, \infty)$  and exponential weights  $\delta = \{\delta_w \in \mathbb{R}\}_{w \in \Theta}$ , which is Fredholm if all  $\delta_w$  are chosen to be positive and sufficiently small. Fixing suitable bundle metrics, the

formal adjoint

$$\dot{\mathbf{D}}^* : W^{k,p,\delta}(\dot{F}) \rightarrow W^{k-1,p,\delta}(\dot{E})$$

is then also Fredholm and is an anti-Cauchy-Riemann type operator on  $\dot{F}$  such that by Proposition 3.14,

$$W^{k-1,p,-\delta}(\dot{F}) = \text{im } \dot{\mathbf{D}} \oplus \ker \dot{\mathbf{D}}^*,$$

allowing us to identify  $\text{coker } \dot{\mathbf{D}}$  with the  $L^2$ -orthogonal complement  $\ker \dot{\mathbf{D}}^* \subset L^{q,\delta}(\dot{F}) \cap W^{k-1,p,-\delta}(\dot{F})$  of  $\text{im } \dot{\mathbf{D}} \subset L^{p,-\delta}(\dot{F}) \subset W^{k-1,p,-\delta}(\dot{F})$  for  $1/p + 1/q = 1$ . Let

$$\Pi : W^{k-1,p,-\delta}(\dot{F}) \rightarrow \ker \dot{\mathbf{D}}^*$$

denote the resulting projection along  $\text{im } \dot{\mathbf{D}}$ . Recall from Definition 3.10 the open and dense set  $\mathcal{CR}_{\mathbb{R}}^*(E)$  of Cauchy-Riemann type operators whose kernels and cokernels are guaranteed to be totally real.

**Lemma 5.4.** *For any open subset  $\mathcal{U} \subset \dot{\Sigma}$ , assume  $V \subset \Gamma(\text{Hom}_{\mathbb{R}}(\dot{E}, \dot{F}))$  is a linear subspace satisfying the following conditions:*

- (1) *All  $\Phi \in V$  have compact support in  $\mathcal{U}$ .*
- (2) *For every  $z \in \mathcal{U}$  and  $\Phi_0 \in \text{Hom}_{\mathbb{R}}(\dot{E}_z, \dot{F}_z)$ , there exists a fixed  $\Phi \in \Gamma(\text{Hom}_{\mathbb{R}}(\dot{E}, \dot{F}))$  satisfying  $\Phi(z) = \Phi_0$  such that for every neighborhood  $\mathcal{U}' \subset \mathcal{U}$  of  $z$ , there also exists a smooth compactly supported function  $\beta : \mathcal{U}' \rightarrow [0, 1]$  with  $\beta(z) = 1$  and  $\beta\Phi \in V$ .*

*Then for any  $\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}^*(E)$ , the linear map*

$$\mathbf{L} : V \rightarrow \text{Hom}_{\mathbb{R}}(\ker \dot{\mathbf{D}}, \ker \dot{\mathbf{D}}^*)$$

*defined by  $\mathbf{L}(\Phi)\eta = \Pi(\Phi\eta)$  is surjective.*

*Proof.* Fix bases  $\eta_1, \dots, \eta_p \in \ker \dot{\mathbf{D}}$  and  $\xi_1, \dots, \xi_q \in \ker \dot{\mathbf{D}}^*$ , which are each complex-linearly independent since  $\dot{\mathbf{D}} \in \mathcal{CR}_{\mathbb{R}}^*(\dot{E})$ . Note that  $L^2$ -products of sections in  $W^{k-1,p,-\delta}(\dot{F})$  with elements of  $\ker \dot{\mathbf{D}}^*$  are always well defined since  $\ker \dot{\mathbf{D}}^* \subset L^{q,\delta}(\dot{F})$  for  $1/p + 1/q = 1$  by Lemma 3.13; in particular, the  $L^2$ -product has a well-defined restriction to  $\ker \dot{\mathbf{D}}^*$  itself. By the orthogonality of  $\text{im } \dot{\mathbf{D}} = \ker \Pi$  to  $\ker \dot{\mathbf{D}}^*$ , we then have

$$\langle \xi_j, \mathbf{L}(\Phi)\eta_i \rangle_{L^2} = \langle \xi_j, \Phi\eta_i \rangle_{L^2} \quad \text{for all } i = 1, \dots, p, j = 1, \dots, q,$$

and these matrix elements determine  $\mathbf{L}(\Phi) : \ker \dot{\mathbf{D}} \rightarrow \ker \dot{\mathbf{D}}^*$ . Now if  $\mathbf{L}$  is not surjective, there exists a nontrivial linear map  $\Psi : \ker \dot{\mathbf{D}} \rightarrow \ker \dot{\mathbf{D}}^*$  which is “orthogonal” to every  $\mathbf{L}(\Phi)$  in the sense that its matrix elements  $\Psi^{ij} := \langle \xi_j, \Psi\eta_i \rangle_{L^2} \in \mathbb{R}$  satisfy

$$(5.5) \quad \sum_{i,j} \Psi^{ij} \langle \xi_j, \Phi\eta_i \rangle_{L^2} = 0$$

for every  $\Phi \in V$ . Denoting the measure on  $\dot{\Sigma}$  by  $d\mu(z)$  and the Hermitian inner product on  $F$  by  $\langle \cdot, \cdot \rangle_F$ , we can rewrite (5.5) as

$$\begin{aligned} 0 &= \sum_{i,j} \Psi^{ij} \int_{\mathcal{U}} \text{Re} \langle \xi_j(z), \Phi(z)\eta_i(z) \rangle_F d\mu(z) \\ &= \int_{\mathcal{U}} \text{Re} \langle \cdot, \cdot \rangle_F \circ (\mathbf{1} \otimes \Phi) \circ \left( \sum_{i,j} \Psi^{ij} \xi_j(z) \otimes \eta_i(z) \right) d\mu(z), \end{aligned}$$

where  $\sum_{i,j} \Psi^{ij} \xi_j \otimes \eta_i$  is regarded as a section of  $F \otimes_{\mathbb{R}} E$ . Since the  $\Psi^{ij}$  are not all zero by assumption, this section is nonzero on an open and dense subset by Corollary 5.2. Choosing a point  $z \in \mathcal{U}$  where it is nonzero, Lemma 5.5 below then provides a linear map  $\Phi_0 : E_z \rightarrow F_z$  such that the integrand is positive near  $z$  for any  $\Phi \in V$  satisfying  $\Phi(z) = \Phi_0$ , and we can then make the entire integral positive after multiplying  $\Phi$  by smooth bump functions with sufficiently small support.  $\square$

We used:

**Lemma 5.5.** *Suppose  $V$  and  $W$  are real finite-dimensional vector spaces,  $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$  is an inner product on  $V$ , and  $T \in V \otimes W$  is nonzero. Then there exists a linear map  $\Phi : W \rightarrow V$  such that  $\langle \cdot, \cdot \rangle \circ (\mathbb{1} \otimes \Phi)(T) > 0$ .*

*Proof.* Choosing a basis  $w_1, \dots, w_n$  of  $W$ , we have  $T = \sum_{j=1}^n v_j \otimes w_j$  for unique vectors  $v_1, \dots, v_n \in V$  that do not all vanish since  $T \neq 0$ . Choosing  $\Phi : W \rightarrow V$  such that  $\Phi(w_j) = v_j$  for all  $j$  then gives  $\langle \cdot, \cdot \rangle \circ (\mathbb{1} \otimes \Phi)(T) = \sum_j \langle v_j, v_j \rangle > 0$ .  $\square$

## 6. A SARD-SMALE ARGUMENT

We are now in a position to prove Theorem D. The main idea behind the proof is standard, though some details are less so: we will define a suitable Banach manifold of perturbed almost complex structures, producing a universal moduli space and a projection to the space of perturbed data whose regular values have the property stated in the theorem. The hard part is of course to prove that the universal moduli space is a smooth Banach manifold—this follows from the implicit function theorem after proving that some version of the operator defined in (3.23) is surjective, and that is where Lemma 5.4 from the previous section is needed.

We start by using the Floer  $C_\varepsilon$ -topology (cf. [Flo88, §5]) to define spaces of perturbed data. Fix a sequence of positive numbers  $\varepsilon_\nu \rightarrow 0$  and, given a closed Riemann surface  $(\Sigma, j)$  with complex vector bundles  $E \rightarrow \Sigma$  and  $F := \overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, E)$ , define the separable Banach space

$$\mathcal{A}_\varepsilon(E) := \{A \in \Gamma(\text{Hom}_{\mathbb{R}}(E, F)) \mid \|A\|_{C_\varepsilon} < \infty\},$$

where

$$\|A\|_{C_\varepsilon} := \sum_{\nu=0}^{\infty} \varepsilon_\nu \|A\|_{C^\nu}.$$

If  $J_{\text{ref}} \in \mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$ , we can view the “tangent space”  $T_{J_{\text{ref}}} \mathcal{J}(J, \omega; \mathcal{U}, J_{\text{fix}})$  as the space of smooth  $J_{\text{ref}}$ -antilinear bundle maps  $Y \in \Gamma(\overline{\text{End}}_{\mathbb{C}}(TM, J_{\text{ref}}))$  that vanish outside  $\mathcal{U}$  and satisfy  $\omega(\cdot, Y\cdot) + \omega(Y\cdot, \cdot) \equiv 0$ , and there is a natural embedding

$$Y \mapsto J_Y := \left( \mathbb{1} + \frac{1}{2} J_{\text{ref}} Y \right) J_{\text{ref}} \left( \mathbb{1} + \frac{1}{2} J_{\text{ref}} Y \right)^{-1}$$

which takes a  $C^0$ -small neighborhood of 0 in  $T_{J_{\text{ref}}} \mathcal{J}(J, \omega; \mathcal{U}, J_{\text{fix}})$  homeomorphically to a neighborhood of  $J_{\text{ref}}$  in  $\mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$ . This allows us to define for any  $\varepsilon > 0$  sufficiently small a smooth, separable and metrizable Banach manifold

$$\mathcal{J}_\varepsilon := \{J_Y \mid Y \in T_{J_{\text{ref}}} \mathcal{J}(J, \omega; \mathcal{U}, J_{\text{fix}}), \|Y\|_{C_\varepsilon} < \varepsilon\}$$



which embeds continuously into  $\mathcal{J}(J, \omega; \mathcal{U}, J_{\text{fix}})$  and contains arbitrarily  $C^\infty$ -small perturbations of  $J_{\text{ref}}$ .

Recall that a point  $z \in \Sigma$  in the domain of a smooth map  $v : \Sigma \rightarrow M$  is called an **injective point** if  $dv(z) : T_z \Sigma \rightarrow T_{v(z)} M$  is injective and  $\{z\} = v^{-1}(v(z))$ . For a simple  $J$ -holomorphic curve, the complement of the set of injective points is a discrete set.

**Lemma 6.1.** *Assume  $J \in \mathcal{J}_\varepsilon$ ,  $v : (\Sigma, j) \rightarrow (M, J)$  is a simple  $J$ -holomorphic curve with generalized normal bundle  $N_v \subset v^* TM$  and normal Cauchy-Riemann operator  $\mathbf{D}_v^N \in \mathcal{CR}_\mathbb{R}(N_v)$ , and  $A \in \mathcal{A}_\varepsilon(N_v)$  has support contained in the set of injective points in  $v^{-1}(\mathcal{U})$ . Then there exists a smooth family of almost complex structures*

$$\{J_t \in \mathcal{J}_\varepsilon\}_{t \in (-\varepsilon, \varepsilon)}$$

*such that  $J_0 = J$ ,  $J_t(v(z)) = J(v(z))$  for all  $t$  and  $z$ , and the resulting family of normal Cauchy-Riemann operators  $\mathbf{D}_{v,t}^N \in \mathcal{CR}_\mathbb{R}(N_v)$  for  $v$  defined with respect to  $J_t$  satisfies*

$$\partial_t \mathbf{D}_{v,t}^N|_{t=0} = A.$$

*Proof.* If  $\{J_t\}$  is any smooth path in  $\mathcal{J}_\varepsilon$  with  $J_0 = J$ ,  $J_t(v) \equiv J(v)$  for all  $t$  and  $Y := \partial_t J|_{t=0}$ , then  $Y(v) \equiv 0$ , hence  $\nabla Y$  is well defined along  $v$  independently of any connection. For  $\eta \in \Gamma(N_v)$ , let us write  $\nabla_\eta Y$  in block form as

$$\nabla_\eta Y = \begin{pmatrix} \nabla_\eta^T Y & \nabla_\eta^{TN} Y \\ \nabla_\eta^{NT} Y & \nabla_\eta^N Y \end{pmatrix} \in \Gamma(\overline{\text{End}}_\mathbb{C}(v^* TM, J))$$

with respect to the tangent-normal decomposition  $v^* TM = T_v \oplus N_v$ . Choosing  $N_v$  for convenience to be the  $\omega$ -symplectic orthogonal complement of  $T_v$ , the fact that  $J_t$  is always  $\omega$ -compatible then translates into conditions that constrain  $\nabla_\eta^T Y$  and  $\nabla_\eta^N Y$  separately and another condition that determines  $\nabla_\eta^{TN} Y$  in terms of  $\nabla_\eta^{NT} Y$ , namely

$$\omega((\nabla_\eta^{NT} Y)v, w) + \omega(v, (\nabla_\eta^{TN} Y)w) = 0$$

for all  $(v, w) \in T_v \oplus N_v$ . This means that  $\omega$ -compatibility does not prevent us from freely choosing  $\nabla_\eta^{NT} Y$  so long as we (1) do not mind  $\nabla_\eta^{TN} Y$  being determined by this choice, and (2) do this only in regions where  $v$  has no double points, so that the splitting of  $TM$  into  $T_v \oplus N_v$  is unambiguous. Now using the definition of the normal Cauchy-Riemann operator, one computes that for any  $\eta \in \Gamma(N_v)$ ,

$$\partial_t \mathbf{D}_{v,t}^N|_{t=0} = \nabla_\eta^{NT} Y \circ Tv \circ j.$$

On a region where  $v$  has neither critical points nor double points and its image lies in the perturbation domain  $\mathcal{U}$ , we can therefore choose the normal derivatives of  $Y$  along  $v$  to make the above expression match  $A$ .  $\square$

We recall next from Definition 3.10 the space of Cauchy-Riemann operators whose complex-antilinear parts are invertible on some fiber.

**Lemma 6.2.** *There exists a Baire subset  $\mathcal{J}_\varepsilon^{\text{reg}} \subset \mathcal{J}_\varepsilon$  such that for all  $J \in \mathcal{J}_\varepsilon^{\text{reg}}$  and all  $g, m \geq 0$ ,  $\ell_1, \dots, \ell_m \geq 1$  and  $A \in H_2(M)$ , every simple curve  $v \in \mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m)$  passing through  $\mathcal{U}$  has  $\mathbf{D}_v^N \in \mathcal{CR}_\mathbb{R}^*(N_v)$ .*

*Proof.* Abbreviate  $\ell := (\ell_1, \dots, \ell_m)$  and define

$$\mathcal{M}_{g,m+1}^*(A, J; \ell) \subset \mathcal{M}_{g,m+1}(A, J)$$

as the space of curves  $v : (\Sigma, j) \rightarrow (M, J)$  with marked points  $\zeta_1, \dots, \zeta_{m+1}$  such that  $\{\zeta_1, \dots, \zeta_m\}$  is the set of critical points of  $v$ , satisfying  $\text{ord}(dv; \zeta_i) = \ell_i$  for  $i = 1, \dots, m$ , and  $\zeta_{m+1}$  is an injective point with  $v(\zeta_{m+1}) \in \mathcal{U}$ . For each pair of integers  $k, q \geq 1$ , we then define a subspace

$$\mathcal{M}_{g,m+1}^*(A, J; \ell, k, q) \subset \mathcal{M}_{g,m+1}^*(A, J; \ell)$$

via the following constraint. Abbreviating the complex vector bundles

$$E := N_v \quad \text{and} \quad F := \overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, N_v)$$

over  $\Sigma$ , define the fiber bundle

$$\overline{\text{Hom}}_{\mathbb{C}}^k(E, F) \subset \overline{\text{Hom}}_{\mathbb{C}}(E, F)$$

whose fiber over  $z \in \Sigma$  is the submanifold of complex-antilinear maps  $A \in \overline{\text{Hom}}_{\mathbb{C}}(E_z, F_z)$  with  $\dim_{\mathbb{C}} \ker A = \dim_{\mathbb{C}} \text{coker } A = k$ . These fibers have complex codimension  $k^2$  in  $\overline{\text{Hom}}_{\mathbb{C}}(E_z, F_z)$ , hence  $\overline{\text{Hom}}_{\mathbb{C}}^k(E, F)$  is a submanifold of  $\overline{\text{Hom}}_{\mathbb{C}}(E, F)$  with real codimension  $2k^2$ . The curve  $v$  determines a natural section of  $\overline{\text{Hom}}_{\mathbb{C}}(E, F)$ , namely the antilinear part of the normal Cauchy-Riemann operator,

$$\mathbf{D}_{v, \overline{\mathbb{C}}}^N := \frac{1}{2} (\mathbf{D}_v^N + J \circ \mathbf{D}_v^N \circ J) : \Sigma \rightarrow \overline{\text{Hom}}_{\mathbb{C}}(E, F),$$

and we define  $v$  to be an element of  $\mathcal{M}_{g,m+1}^*(A, J; \ell, k, q)$  if and only if this section meets  $\overline{\text{Hom}}_{\mathbb{C}}^k(E, F)$  at  $\zeta_{m+1}$  with a tangency of order  $q$ , i.e. its  $q$ -jet matches that of a map with image in  $\overline{\text{Hom}}_{\mathbb{C}}^k(E, F)$ .

The two spaces defined above have corresponding universal moduli spaces

$$\mathcal{U}^*(\mathcal{J}_{\varepsilon}; \ell) \quad \text{and} \quad \mathcal{U}^*(\mathcal{J}_{\varepsilon}; \ell, k, q),$$

each consisting of pairs  $(v, J)$  where  $J \in \mathcal{J}_{\varepsilon}$  and  $v$  belongs to either  $\mathcal{M}_{g,m+1}^*(A, J; \ell)$  or  $\mathcal{M}_{g,m+1}^*(A, J; \ell, k, q)$  respectively. The arguments of Appendix A show that  $\mathcal{U}^*(\mathcal{J}_{\varepsilon}; \ell)$  is a smooth, metrizable and separable Banach manifold; indeed, for any element  $(v_0, J_0) \in \mathcal{U}^*(\mathcal{J}_{\varepsilon}; \ell)$  represented by  $v_0 : (\Sigma, j_0) \rightarrow (M, J_0)$ , one can identify a neighborhood of  $(v_0, J_0)$  in  $\mathcal{U}^*(\mathcal{J}_{\varepsilon}; \ell)$  with some finite-codimensional smooth submanifold

$$X \subset \bar{\partial}^{-1}(0) \subset \mathcal{T} \times \mathcal{B} \times \mathcal{J}_{\varepsilon}$$

in the smooth zero-set of the nonlinear Cauchy-Riemann operator  $\bar{\partial}$  defined on the product of  $\mathcal{J}_{\varepsilon}$  with  $\mathcal{B} := W^{k,p}(\Sigma, M)$  for  $p \in (1, \infty)$  and  $k \in \mathbb{N}$  sufficiently large, and a Teichmüller slice  $\mathcal{T}$  through  $j_0$  (see Appendix A). The submanifold  $X \subset \bar{\partial}^{-1}(0)$  is determined by constraints on the jet evaluation map that produce the correct critical orders for the first  $m$  marked points; the only difference between this and the setup in the appendix is that since there is one extra marked point, the Teichmüller slice is larger by 2 dimensions. Now we claim that the intersection of this neighborhood of  $(v_0, J_0)$  with  $\mathcal{U}^*(\mathcal{J}_{\varepsilon}; \ell, k, q)$  defines a smooth submanifold of  $X$  with real codimension

$$(6.1) \quad \text{codim } \mathcal{U}^*(\mathcal{J}_{\varepsilon}; \ell, k, q) = k^2(q+1)(q+2).$$

Indeed, the condition defining  $\mathcal{U}^*(\mathcal{J}_\varepsilon; \ell, k, q)$  constrains the  $q$ -jet of  $\mathbf{D}_{v, \overline{\mathbb{C}}}^N$  at  $\zeta_{m+1}$  to intersect a submanifold of precisely this dimension, i.e. the product of  $\text{codim}_{\mathbb{R}} \overline{\text{Hom}}_{\mathbb{C}}^k(E, F) = 2k^2$  with  $1 + 2 + \dots + q + (q+1)$ , which is the number of distinct multi-indices up to order  $q$  for a function of two real variables. This intersection is also transverse: to see this, observe that by Lemma 6.1,  $T_{(j,v,J)}X$  contains elements of the form  $(0, 0, Y)$ , where  $Y \in T_J \mathcal{J}_\varepsilon$  can be chosen to produce arbitrary perturbations to  $\mathbf{D}_v^N$  of class  $C_\varepsilon$  near  $v(\zeta_{m+1})$ . In particular,  $Y$  can be chosen to perturb  $\mathbf{D}_{v, \overline{\mathbb{C}}}^N$  by a section with any desired Taylor polynomial of some finite but large degree at  $\zeta_{m+1}$ . With the claim established, we can apply the Sard-Smale theorem to the projection

$$\mathcal{U}^*(\mathcal{J}_\varepsilon; \ell, k, q) \rightarrow \mathcal{J}_\varepsilon : (v, J) \mapsto J,$$

and thus find a Baire subset of  $\mathcal{J}_\varepsilon$  for which  $\mathcal{M}_{g,m+1}^*(A, J; \ell)$  and  $\mathcal{M}_{g,m+1}^*(A, J; \ell, k, q)$  are both smooth finite-dimensional manifolds and the codimension of the latter in the former is given by (6.1). This means

$$\dim \mathcal{M}_{g,m+1}^*(A, J; \ell, k, q) = \text{vir-dim } \mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m) + 2 - k^2(q+1)(q+2)$$

will always become negative when  $q$  is chosen large enough, and will therefore be empty for  $J$  in our Baire subset. Finally, observe that any simple curve  $v \in \mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m)$  that passes through  $\mathcal{U}$  and has  $\mathbf{D}_v^N \notin \mathcal{CR}_{\mathbb{R}}^*(N_v)$  can be regarded as an element of the space  $\mathcal{M}_{g,m+1}^*(A, J; \ell, k, q)$  for some  $k \geq 1$  and arbitrarily large  $q \geq 1$  after adding an extra marked point somewhere in  $v^{-1}(\mathcal{U})$ .  $\square$

*Remark 6.3.* The proof above would work equally well to find generic families of almost complex structures depending on finitely many parameters, since the codimension in (6.1) can be made arbitrarily large by choosing  $q$  larger. This detail is important for the bifurcation theory discussed in §2.4.

We now proceed toward the proof of Theorem D. For each of the choices of data in the statement of Theorem D, we define two universal moduli spaces

$$\mathcal{U}^d(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m), \quad \mathcal{U}^d(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m; \mathbf{k}, \mathbf{c}),$$

which consist of pairs  $(u, J)$  with  $J \in \mathcal{J}_\varepsilon$ , where in the first case  $u$  is assumed to belong to  $\mathcal{M}_{\mathbf{b}, G}^d(\mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m))$ , and in the second  $\mathcal{M}_{\mathbf{b}, G}^d(\mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m); \mathbf{k}, \mathbf{c})$ . Writing  $u = v \circ \varphi$  for the underlying simple curve  $v$ , we further impose the open condition

$$\mathbf{D}_v^N \in \mathcal{CR}_{\mathbb{R}}^*(N_v)$$

on both of these spaces, so that the kernel and cokernel of any pullback of  $\mathbf{D}_v^N$  are guaranteed by Lemma 3.11 to be totally real. The first space is easily seen to be a smooth separable Banach manifold: indeed, suppose  $(u_0, J_0) \in \mathcal{U}^d(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m)$ , where  $u_0 = v_0 \circ \varphi_0$  for some simple  $J_0$ -holomorphic curve  $v_0 : (\Sigma, j_0) \rightarrow (M, J_0)$  and  $d$ -fold branched cover  $\varphi_0 : (\Sigma', j'_0) \rightarrow (\Sigma, j_0)$  representing an element of  $\mathcal{M}_{\mathbf{b}}^d(j_0)$  with generalized automorphism group  $G$ . Then  $(v_0, J_0)$  lives in the smooth universal moduli space  $\mathcal{U}^*(\mathcal{J}_\varepsilon)$  defined in Appendix A, more specifically in the finite-codimensional submanifold

$$\mathcal{U}^*(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m) \subset \mathcal{U}^*(\mathcal{J}_\varepsilon)$$

of this space defined by the condition that the  $i$ th marked point should have critical order  $\ell_i$  and curves are immersed everywhere else. As in the proof of Lemma 6.2, a neighborhood of  $(v_0, J_0)$  in  $\mathcal{U}^*(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m)$  can thus be identified with a smooth finite-codimensional submanifold

$$X \subset \bar{\partial}^{-1}(0) \subset \mathcal{T} \times \mathcal{B} \times \mathcal{J}_\varepsilon$$

of the zero-set of the nonlinear Cauchy-Riemann operator. In parallel with this, we can parametrize a neighborhood of  $\varphi_0$  in  $\mathcal{M}_\mathbf{b}^d(j_0)$  as explained in Examples 3.4 and 3.6, meaning that if  $\Theta = \{w_1, \dots, w_r\} \subset \Sigma$  is the set of critical values of  $\varphi_0$ , we choose a smooth family of diffeomorphisms  $\psi_\tau : \Sigma \rightarrow \Sigma$  parametrized by  $\tau \in B^{2r}$  which are holomorphic near  $\Theta$  and supported on a slightly larger neighborhood of  $\Theta$  such that  $\psi_0 = \text{Id}$  and

$$B^{2r} \rightarrow \Sigma^{\times r} : \tau \mapsto (\psi_\tau(w_1), \dots, \psi_\tau(w_r))$$

is a diffeomorphism onto an open set. The neighborhood of  $(u_0, J_0)$  in  $\mathcal{U}^d(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m)$  can now be identified with  $B^{2r} \times X$  by associating to each  $(\tau, (j, v, J)) \in B^{2r} \times X$  the curve  $v \circ (\psi_\tau \circ \varphi_0)$ . Following Example 3.6, this fits into the general picture from §3 of a parametrized family of bundles with Cauchy-Riemann operators, where the parameter space is the Banach manifold

$$P = B^{2r} \times X \subset B^{2r} \times \bar{\partial}^{-1}(0) \subset B^{2r} \times (\mathcal{T} \times \mathcal{B} \times \mathcal{J}_\varepsilon),$$

and for any  $\tau = (\sigma, (j, v, J)) \in P$  we have the data

- $\psi_\tau := \psi_\sigma$ ,
- $j_\tau := j$ ,
- $(E_\tau, J_\tau) := (N_v, J)$ ,
- $\mathbf{D}_\tau := \mathbf{D}_v^N$ .

We should pause a moment to discuss why the generalized normal bundles  $N_v$  form a smooth family as  $(j, v, J) \in X$  varies—this depends crucially on the critical point constraints satisfied by  $v$ . Assuming  $N_v \subset v^*TM$  is always defined as the symplectic orthogonal complement of  $T_v \subset v^*TM$  with  $T_v := \text{im } dv$  away from critical points, let us recall from [Wen10] how the latter is defined at critical points. We have a smooth family of bundles  $v^*TM$  carrying linearized Cauchy-Riemann operators  $\mathbf{D}_v$ , whose complex-linear parts  $\mathbf{D}_v^C$  define a smooth family of holomorphic structures on  $v^*TM$ . The crucial observation is then that  $dv \in \Gamma(\text{Hom}_\mathbb{C}(T\Sigma, v^*TM))$  is always a holomorphic section with respect to the holomorphic bundle structures on  $v^*TM$  and  $T\Sigma$ , so choosing a smooth family of holomorphic trivializations and holomorphic coordinates near the  $i$ th marked point, each  $dv$  is represented by some holomorphic function of the form

$$f_v^{(i)} : \mathbb{D} \rightarrow \mathbb{C}^m, \quad f_v^{(i)}(z) = z^{\ell_i} g_v^{(i)}(z),$$

where  $g_v^{(i)} : \mathbb{D} \rightarrow \mathbb{C}^m$  is another family of holomorphic functions which depend smoothly on  $(j, v, J) \in X$  but also are nonzero at 0. The main point here is that the critical orders  $\ell_i$  do not vary with  $v$ . The span of  $g_v^{(i)}(0)$  thus defines the fibers of  $T_v$  near each critical point, so we deduce smooth dependence of  $T_v$  on  $(j, v, J) \in X$ .

If  $(u_0, J_0) \in \mathcal{U}^d(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m; \mathbf{k}, \mathbf{c})$ , then using the setup in §3.5, we now find a regular branched cover  $\hat{\varphi}_0 : (\hat{\Sigma}, \hat{j}_0) \rightarrow (\Sigma, j_0)$  with automorphism group

$$G := \text{Aut}(\hat{\varphi}_0)$$

and corresponding  $J_0$ -holomorphic curve  $\tilde{u}_0 = v_0 \circ \hat{\varphi}_0$ , with a smooth map

$$\mathbf{F} : B^{2r} \times X \rightarrow \text{Hom}_G(\ker \dot{\mathbf{D}}_{\tilde{u}_0}^N, \ker(\dot{\mathbf{D}}_{\tilde{u}_0}^N)^*)$$

whose zero-set is a neighborhood of  $(u_0, J_0)$  in  $\mathcal{W}^d(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m; \mathbf{k}, \mathbf{c})$ . Here the normal operator  $\dot{\mathbf{D}}_{\tilde{u}_0}^N$  is defined over the punctured domain  $\hat{\Sigma}$  with negative exponential weights close to zero, and its formal adjoint  $(\dot{\mathbf{D}}_{\tilde{u}_0}^N)^*$  has correspondingly positive (but small) exponential weights.

**Lemma 6.4.** *The linearization of  $\mathbf{F}$  at  $(0, (j_0, v_0, J_0))$  is surjective.*

*Proof.* By Lemma 6.1, the tangent space to  $B^{2r} \times X$  at  $(0, (j_0, v_0, J_0))$  contains a large space of elements of the form  $(0, (0, 0, Y))$  for  $Y \in T_{J_0}\mathcal{J}_\varepsilon$  which can be chosen to realize any perturbation in  $\mathbf{D}_{v_0}^N$  of the form

$$\mathbf{D}_{v_0}^N \rightsquigarrow \mathbf{D}_{v_0}^N + A_Y,$$

where  $A_Y$  is a zeroth-order term of class  $C_\varepsilon$  with support in  $v_0^{-1}(\mathcal{U})$  away from the critical and double points of  $v_0$ . The resulting change in  $\mathbf{D}_{\tilde{u}_0}$  is

$$\mathbf{D}_{\tilde{u}_0} \rightsquigarrow \mathbf{D}_{\tilde{u}_0} + \tilde{\varphi}_0^* A_Y,$$

hence differentiating  $\mathbf{F}$  in the direction  $(0, (0, 0, Y))$  produces a  $G$ -equivariant linear map  $\mathbf{L}(Y) : \ker \dot{\mathbf{D}}_{\tilde{u}_0}^N \rightarrow \ker(\dot{\mathbf{D}}_{\tilde{u}_0}^N)^*$  given by (3.23), namely

$$\mathbf{L}(Y)\eta = \Pi((\hat{\varphi}_0^* A_Y)\eta),$$

in terms of the projection

$$W^{k-1, p, -\delta'}(N_{\tilde{u}_0}) = \text{im}(\dot{\mathbf{D}}_{\tilde{u}_0}^N) \oplus \ker(\dot{\mathbf{D}}_{\tilde{u}_0}^N)^* \rightarrow \ker(\dot{\mathbf{D}}_{\tilde{u}_0}^N)^*.$$

We claim that  $Y$  can be chosen to make  $\mathbf{L}(Y)$  equal to any given element

$$\Psi \in \text{Hom}_G(\ker \dot{\mathbf{D}}_{\tilde{u}_0}^N, \ker(\dot{\mathbf{D}}_{\tilde{u}_0}^N)^*).$$

Indeed, let us abbreviate  $E = N_{v_0}$  and  $F = \overline{\text{Hom}}_{\mathbb{C}}(T\Sigma_0, N_{v_0})$ . Since  $\mathbf{D}_{v_0}^N \in \mathcal{CR}_{\mathbb{R}}^*(E)$  and therefore also  $\dot{\mathbf{D}}_{\tilde{u}_0}^N \in \mathcal{CR}_{\mathbb{R}}^*(\tilde{\varphi}_0^* \dot{E})$ , Lemma 5.4 provides for any given  $\Psi$  a section  $\tilde{A} \in \Gamma(\text{Hom}_{\mathbb{R}}(\tilde{\varphi}_0^* \dot{E}, \tilde{\varphi}_0^* \dot{F}))$  satisfying  $\|\tilde{A}\|_{C_\varepsilon} < \infty$ , with support in some small neighborhood of a point  $z_0 \in \hat{\Sigma}$  where  $\tilde{u}_0(z_0) \in \mathcal{U}$  and  $\varphi_0(z_0)$  is an injective point of  $v_0$ , such that

$$\langle \xi, \tilde{A}\eta \rangle_{L^2} = \langle \xi, \Psi\eta \rangle_{L^2}$$

for all  $\xi \in \ker(\dot{\mathbf{D}}_{\tilde{u}_0}^N)^*$  and  $\eta \in \ker \dot{\mathbf{D}}_{\tilde{u}_0}^N$ . Note that we are free to assume the  $L^2$ -product is invariant under the action of  $G$  via deck transformations. Then since  $\Psi$  is  $G$ -equivariant, we also have for every  $g \in G$ ,

$$\langle \xi, (g\tilde{A})\eta \rangle_{L^2} = \langle g^{-1}\xi, \tilde{A}(g^{-1}\eta) \rangle_{L^2} = \langle g^{-1}\xi, \Psi(g^{-1}\eta) \rangle_{L^2} = \langle g^{-1}\xi, g^{-1}(\Psi\eta) \rangle_{L^2} = \langle \xi, \Psi\eta \rangle_{L^2},$$

implying that the symmetrization  $\tilde{A}_G := \frac{1}{|G|} \sum_{g \in G} g\tilde{A}$  also satisfies

$$\langle \xi, \tilde{A}_G\eta \rangle_{L^2} = \langle \xi, \Psi\eta \rangle_{L^2}$$

for all  $\xi, \eta$ . But the  $G$ -invariance of  $\tilde{A}_G$  implies  $\tilde{A}_G = \tilde{\varphi}_0^* A$  for some  $A = A_Y$  with  $Y \in T_{J_0}\mathcal{J}_\varepsilon$ , so we are done.  $\square$

The lemma implies via the implicit function theorem that

$$\mathcal{U}^d(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m; \mathbf{k}, \mathbf{c}) \subset \mathcal{U}^d(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m)$$

is a smooth Banach submanifold with codimension given by the formula in (3.22). We can then apply the Sard-Smale theorem to the projection

$$\mathcal{U}^d(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m; \mathbf{k}, \mathbf{c}) \rightarrow \mathcal{J}_\varepsilon : (u, J) \mapsto J$$

and thus find a Baire subset in  $\mathcal{J}_\varepsilon$  for which the statement of Theorem D holds so long as we exclude all covers of simple curves  $v$  with  $\mathbf{D}_v^N \notin \mathcal{CR}_\mathbb{R}^*(N_v)$ . Intersecting further with the Baire subset provided by Lemma 6.2, we can assume without loss of generality that no such curves exist.

Since the reference structure  $J_{\text{ref}}$  in the definition of  $\mathcal{J}_\varepsilon$  was arbitrary, this implies in particular that the theorem holds for a *dense* set of  $J$  in  $\mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$ . To turn this into a Baire subset, we now apply the standard Taubes trick (cf. [Wen, §4.4.2]): the idea is to exhaust  $\mathcal{M}_{\mathbf{b}, G}^d(\mathcal{M}_{g, m}(A, J; \ell_1, \dots, \ell_m); \mathbf{k}, \mathbf{c})$  by a countable sequence of compact subsets

$$\mathcal{M}_{\mathbf{b}, G}^d(\mathcal{M}_{g, m}(A, J; \ell_1, \dots, \ell_m); \mathbf{k}, \mathbf{c}; K) \subset \mathcal{M}_{\mathbf{b}, G}^d(\mathcal{M}_{g, m}(A, J; \ell_1, \dots, \ell_m); \mathbf{k}, \mathbf{c})$$

for  $K \in \mathbb{N}$ , each defined as the set of curves  $v \circ \varphi$  that lie at distance at least  $1/K$  separated (with respect to some metric) from any of the closed subsets in the Gromov compactification where  $\varphi$  becomes a nodal branched cover, or two of its critical values run together, or  $v$  becomes a nodal curve, or a multiple cover, or its image escapes from  $\mathcal{U}$ , or its algebraic sum of critical points increases. This can be done in a way that depends continuously on  $J$ , so the subset

$$\mathcal{J}_K^{\text{reg}} \subset \mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$$

consisting of  $J$  for which  $\mathcal{M}_{\mathbf{b}, G}^d(\mathcal{M}_{g, m}(A, J; \ell_1, \dots, \ell_m); \mathbf{k}, \mathbf{c}; K)$  is a submanifold of the correct dimension (meaning it is locally the zero-set of a submersion to a vector space of that dimension) is open, while the Sard-Smale argument above shows that it is also dense. The countable intersection of these open and dense subsets is therefore the Baire subset of  $\mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$  promised by Theorem D, and the proof of the theorem is complete.

## 7. SUPER-RIGIDITY IN DIMENSION FOUR

We now prove the 4-dimensional case of Theorem A, using intersection-theoretic arguments that are essentially unrelated to the rest of the paper. Throughout this section, assume  $(M, J)$  is an almost complex manifold with

$$\dim M = 4.$$

The genus zero case is an “automatic” phenomenon, i.e. it does not require any genericity condition except for ensuring that the index 0 simple curve is immersed:

**Proposition 7.1.** *Every simple immersed  $J$ -holomorphic sphere  $v : (S^2, i) \rightarrow (M, J)$  of index 0 in an almost complex 4-manifold is super-rigid.*

*Proof.* Assume  $\varphi : (\Sigma', j') \rightarrow (S^2, i)$  is a  $d$ -fold branched cover and  $u = v \circ \varphi$ . Since  $v$  is immersed, the Riemann-Roch formula implies

$$0 = \text{ind}(v) = \text{ind } \mathbf{D}_v^N = \chi(S^2) + 2c_1(N_v),$$

hence  $c_1(N_v) = -1$ . Then  $c_1(N_u) = c_1(\varphi^*N_v) = -d$ , so if  $\eta \in \ker \mathbf{D}_u^N$  is nontrivial, its algebraic count of zeroes is negative, violating the similarity principle.  $\square$

For the genus one case, we use a variant of the “magic trick” proposed by Hutchings [Hut] in the context of Embedded Contact Homology.

**Proposition 7.2.** *A simple immersed  $J$ -holomorphic torus  $v : (\mathbb{T}^2, j) \rightarrow (M, J)$  of index 0 in an almost complex 4-manifold is super-rigid if and only if all its unbranched covers are Fredholm regular.*

*Proof.* We will assume for most of the proof that  $v : (\Sigma, j) \rightarrow (M, J)$  has unspecified genus  $g \geq 1$ . Since  $v$  is immersed with index 0, it is regular if and only if its normal Cauchy-Riemann operator  $\mathbf{D}_v^N$  is injective, so given this and the assumption that the same holds for all unbranched covers  $u = v \circ \varphi$ , we need to show that  $\mathbf{D}_u^N$  is also injective for  $u = v \circ \varphi$  where  $\varphi : (\Sigma', j') \rightarrow (\Sigma, j)$  is *any* holomorphic branched cover. We will prove this by induction on the degree  $d := \deg(\varphi)$ , thus assume it is true for all covers up to degree  $d - 1$ . Note that since  $\text{ind}(v) = 0$ , we have

$$(7.1) \quad \text{ind } \mathbf{D}_v^N = \chi(\Sigma) + 2c_1(N_v) = 0,$$

and if  $\varphi$  has branch points, then  $g' > 1$  by the Riemann-Hurwitz formula.

By the construction in the proof of Proposition 1.3, one can endow the total space of the normal bundle  $\pi : N_v \rightarrow \Sigma$  with an almost complex structure  $J_N$  such that  $J_N$ -holomorphic curves  $u_\eta : (S, i) \rightarrow (N_v, J_N)$  correspond to sections  $\eta \in \ker \mathbf{D}_{v \circ \psi}^N$  along holomorphic branched covers  $\psi = \pi \circ u_\eta : (S, i) \rightarrow (\Sigma, j)$ . If  $\ker \mathbf{D}_u^N$  contains a nontrivial element  $\eta$ , the inductive hypothesis implies that the corresponding  $J_N$ -holomorphic curve  $u_\eta$  is somewhere injective. We can view  $v$  itself as a  $J_N$ -holomorphic embedding into  $N_v$ , and  $u_\eta$  is homologous to its  $d$ -fold cover, so applying the adjunction formula to both  $u_\eta$  and  $v$  as  $J_N$ -holomorphic curves in  $N_v$ ,

$$\begin{aligned} u_\eta \bullet u_\eta &= 2\delta(u_\eta) + c_1(u_\eta^*TN_v) - \chi(\Sigma') = 2\delta(u_\eta) + d \cdot c_1(v^*TN_v) - \chi(\Sigma') \\ &= d^2(v \bullet v) = d^2 \cdot c_1(N_v) = d^2 \cdot c_1(v^*TN_v) - d^2 \cdot \chi(\Sigma), \end{aligned}$$

where  $\delta(u_\eta) \geq 0$  denotes the algebraic count of double points and critical points of  $u_\eta$ . Solving for  $\delta(u_\eta)$  and plugging in (7.1) to compute  $c_1(v^*TN_v) = \chi(\Sigma) + c_1(N_v) = \frac{1}{2}\chi(\Sigma) = 1 - g$ , we have

$$\begin{aligned} 2\delta(u_\eta) &= d(d-1) \cdot c_1(v^*TN_v) - d^2 \cdot \chi(\Sigma) + \chi(\Sigma') \\ &= d(d-1)(1-g) - 2d^2(1-g) + 2 - 2g' = d(d+1)(g-1) - 2(g'-1) \end{aligned}$$

Plugging in  $g = 1$  and the fact that  $g' > 1$ , this gives a contradiction since  $\delta(u_\eta)$  cannot be negative.  $\square$

*Remark 7.3.* In the spirit of §2.4, the two results above show that the story of super-rigidity and bifurcations is simpler in dimension four. In the genus zero case bifurcations can be avoided altogether: since having a critical point is a codimension 2 condition (see Appendix A), index 0 simple curves for generic 1-parameter families of almost complex structures can be assumed immersed, and therefore super-rigid by Prop. 7.1. This is no longer true in the genus one case since regularity of some unbranched cover might fail under a generic homotopy, producing the birth-death or degree-doubling bifurcations in



[Tau96a], but Prop. 7.2 implies that this is the only danger—the only bifurcations that can happen involve unbranched covers with  $g' = 1$  and  $d \in \{1, 2\}$ , and they are already described in [Tau96a].

*Remark 7.4.* The proof of Prop. 7.2 also gives a partial result for higher genus, though it is not very strong. Suppose  $u$  is a  $d$ -fold cover with genus  $g'$  of a simple index 0 curve  $v$  with genus  $g > 1$  that is regular and immersed. If additionally

$$d(d+1)(g-1) < 2(g'-1),$$

then either  $\mathbf{D}_u^N$  is injective or  $u$  factors through another cover  $u_0$  of  $v$  with smaller degree such that  $\ker \mathbf{D}_{u_0}^N$  is nontrivial.

## APPENDIX A. MODULI SPACES WITH PRESCRIBED ORDERS OF CRITICAL POINTS

The theorem below is well known to experts, but a proof of it is difficult to find in the literature, so we will sketch one here.

Fix a symplectic manifold  $(M, \omega)$  of dimension  $2n$ ,  $n \in \mathbb{N}$ , and suppose  $J \in \mathcal{J}(M, \omega)$ . Recall that if  $(\Sigma, j)$  is a connected Riemann surface and  $u : (\Sigma, j) \rightarrow (M, J)$  is a non-constant  $J$ -holomorphic curve with a critical point  $du(z) = 0$ , then the critical point is isolated and has a well-defined positive **order**,

$$\text{ord}(du; z) \in \mathbb{N},$$

characterized by the property that  $\text{ord}(du; z) = \ell$  if  $z$  is a zero of order  $\ell$  for  $du \in \Gamma(\text{Hom}_{\mathbb{C}}(T\Sigma, u^*TM))$ , where the latter is viewed as a holomorphic section with respect to a natural holomorphic bundle structure on  $u^*TM$  determined by the linearized Cauchy-Riemann operator, see e.g. [Wen10, §3.3]. When  $(\Sigma, j)$  is closed, we denote the resulting algebraic count of critical points by

$$Z(du) := \sum_{\{z \in \Sigma \mid du(z)=0\}} \text{ord}(du; z) \geq 0,$$

and note that it vanishes if and only if  $u$  is immersed. Given integers  $g, m \geq 0$ , a homology class  $A \in H_2(M)$  and a tuple of positive integers  $(\ell_1, \dots, \ell_m)$ , let

$$\mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m) \subset \mathcal{M}_{g,m}(A, J)$$

denote the following subset of the moduli space of unparametrized  $J$ -holomorphic curves homologous to  $A$  with genus  $g$  and  $m$  marked points: a map  $u : (\Sigma, j) \rightarrow (M, J)$  with marked points  $\zeta_1, \dots, \zeta_m \in \Sigma$  representing an element of  $\mathcal{M}_{g,m}(A, J)$  belongs to  $\mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m)$  if and only if it is critical at all marked points,

$$\text{ord}(du; \zeta_j) = \ell_j \quad \text{for } j = 1, \dots, m,$$

and it is immersed everywhere else.

**Proposition A.1.** *Fix an open subset  $\mathcal{U} \subset M$  with compact closure and a compatible almost complex structure  $J_{\text{fix}} \in \mathcal{J}(M, \omega)$ . There exists a Baire subset*

$$\mathcal{J}^{\text{reg}} \subset \mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$$

such that for all  $J \in \mathcal{J}^{\text{reg}}$  and all  $g, m \geq 0$ ,  $A \in H_2(M)$  and  $(\ell_1, \dots, \ell_m) \in \mathbb{N}^m$ , the open subset of  $\mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m)$  consisting of somewhere injective curves that pass through  $\mathcal{U}$  is a smooth manifold with dimension equal to its virtual dimension, where

$$\text{vir-dim } \mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m) = \text{vir-dim } \mathcal{M}_g(A, J) - \sum_{i=1}^m (2n\ell_i - 2).$$

**Corollary A.2.** *For generic compatible  $J$  in any closed symplectic  $2n$ -manifold, all closed, connected and somewhere injective  $J$ -holomorphic curves  $u$  with  $m \geq 0$  critical points satisfy  $\text{ind}(u) \geq 2nZ(du) - 2m$ .*

One well-known consequence of this result is that for generic  $J$ , somewhere injective index 0 curves in almost complex manifolds of dimension at least four are always immersed. Another proof of this is given in [OZ09], though it is analytically somewhat more complicated than the one given below.

It will suffice to prove that the same statement as in Prop. A.1 holds for the slightly larger moduli space

$$\widehat{\mathcal{M}}_{g,m}(A, J; \ell_1, \dots, \ell_m)$$

characterized by the condition  $\text{ord}(du; \zeta_j) \geq \ell_j$  for all  $j = 1, \dots, m$  without requiring  $u$  to be immersed outside the marked points. Indeed,  $\mathcal{M}_{g,m}(A, J; \ell_1, \dots, \ell_m) \subset \widehat{\mathcal{M}}_{g,m}(A, J; \ell_1, \dots, \ell_m)$  is an open subset. We shall borrow from Zehmisch [Zeh15] the notion of *holomorphic jets*: given a point  $p$  in an almost complex manifold  $(M, J)$  and an integer  $r > 0$ , a **holomorphic  $r$ -jet** at  $p$  is an equivalence class of  $J$ -holomorphic curves

$$u : (\mathbb{D}_\epsilon, i) \rightarrow (M, J)$$

with  $u(0) = p$ , where  $(\mathbb{D}_\epsilon, i)$  denotes the  $\epsilon$ -disk in  $\mathbb{C}$ , and two curves are considered equivalent if their partial derivatives at 0 match up to order  $r$ . The nonlinear Cauchy-Riemann equation implies that the holomorphic  $r$ -jet represented by  $u$  is determined by the holomorphic part of its Taylor polynomial of degree  $r$  (see [Wen, Prop. 2.99]), and moreover, every holomorphic Taylor polynomial of degree  $r$  is realizable as the  $r$ -jet of a local  $J$ -holomorphic curve ([Wen, Theorem 2.100]). Thus the space of all holomorphic  $r$ -jets at  $p$  is a real  $2rn$ -dimensional vector space, and the union of these spaces for all  $p \in M$  forms a smooth manifold

$$\text{Jet}_J^r(M)$$

of real dimension  $2n(r+1)$ .

We shall analyze the local structure of  $\widehat{\mathcal{M}}_{g,m}(A, J; \ell_1, \dots, \ell_m)$  following a minor modification of the scheme outlined in [Wen, Chapter 4]. Given  $(\Sigma, j_0, \Theta, u_0)$  representing an element of  $\widehat{\mathcal{M}}_{g,m}(A, J; \ell_1, \dots, \ell_m)$ , with marked points  $\Theta := (\zeta_1, \dots, \zeta_m)$ , choose a **Teichmüller slice** through  $j_0$ : this means a smooth  $(6g - 6 + 2m)$ -dimensional family  $\mathcal{T}$  of complex structures on  $\Sigma$  that includes  $j_0$  and parametrizes a neighborhood of  $[j_0]$  in the Teichmüller space of complex structures modulo diffeomorphisms that are homotopic to the identity and fix  $\Theta$ . The tangent space  $T_{j_0}\mathcal{T}$  is also required to define a closed complement of the image of the canonical Cauchy-Riemann operator on  $T\Sigma$  restricted to the space of vector fields vanishing at  $\Theta$ , cf. [Wen, Definition 4.29]. Moreover, we can arrange for  $\mathcal{T}$  to have the following two properties (cf. [Wen10, Lemmas 3.3 and 3.4]):

- $\mathcal{T}$  is invariant under the action of the group  $\text{Aut}(\Sigma, j_0, \Theta)$  of biholomorphic maps on  $(\Sigma, j_0)$  fixing  $\Theta$ ;
- There exists a neighborhood of  $\Theta$  on which every  $j \in \mathcal{T}$  matches  $j_0$ .

Now let  $r := \max\{\ell_1, \dots, \ell_m\}$ , and choose any  $k \in \mathbb{N}$  and  $p \in (1, \infty)$  such that

$$(A.1) \quad (k - r)p > 2,$$

so the Sobolev embedding theorem implies that functions of class  $W^{k,p}$  on  $\Sigma$  are also in  $C^r$ . We define the Banach manifold

$$\mathcal{B} := W^{k,p}(\Sigma, M)$$

and smooth Banach space bundle  $\mathcal{E} \rightarrow \mathcal{T} \times \mathcal{B}$  with fibers

$$\mathcal{E}_{(j,u)} := W^{k-1,p}(\overline{\text{Hom}_{\mathbb{C}}((T\Sigma, j), (u^*TM, J))}),$$

so that

$$\bar{\partial}_J : \mathcal{T} \times \mathcal{B} \rightarrow \mathcal{E} : (j, u) \mapsto Tu + J \circ Tu \circ j$$

defines a smooth section. We say that  $(\Sigma, j_0, \Theta, u_0)$  is **Fredholm regular** if the linearization

$$D\bar{\partial}_J(j_0, u_0) : T_{j_0}\mathcal{T} \oplus W^{k,p}(u_0^*TM) \rightarrow W^{k-1,p}(\overline{\text{Hom}_{\mathbb{C}}((T\Sigma, j_0), (u_0^*TM, J))})$$

of this section at  $(j_0, u_0)$  is surjective, in which case a neighborhood of  $(j_0, u_0)$  in  $\bar{\partial}_J^{-1}(0)$  is a smooth finite-dimensional manifold, and its quotient by the natural action of  $\text{Aut}(\Sigma, j_0, \Theta)$  can be identified naturally with a neighborhood of  $[(\Sigma, j_0, \Theta, u_0)]$  in  $\mathcal{M}_{g,m}(A, J)$ . To incorporate the critical point condition, fix holomorphic coordinates identifying a neighborhood of each marked point  $\zeta_j$  with the standard unit disk  $(\mathbb{D}, i)$ ; note that this can be done for all  $j \in \mathcal{T}$  at once since they are assumed to match  $j_0$  near  $\Theta$ . Then since  $\mathcal{B}$  has a continuous inclusion into  $C^r(\Sigma, M)$ , there is a well-defined and smooth<sup>5</sup> *jet evaluation map*

$$\text{ev} : \bar{\partial}_J^{-1}(0) \rightarrow \text{Jet}_J^{\ell_1}(M) \times \dots \times \text{Jet}_J^{\ell_m}(M),$$

whose  $i$ th factor for  $i = 1, \dots, m$  is the holomorphic  $\ell_i$ -jet represented by  $u$  in its parametrization by  $(\mathbb{D}, i)$  at  $\zeta_i$ . We will say that  $(\Sigma, j_0, \Theta, u_0)$  is **regular for the constrained moduli space**  $\widehat{\mathcal{M}}_{g,m}(A, J; \ell_1, \dots, \ell_m)$  if it is Fredholm regular and the jet evaluation map is transverse to the submanifold

$$Z \subset \text{Jet}_J^{\ell_1}(M) \times \dots \times \text{Jet}_J^{\ell_m}(M)$$

consisting of  $m$ -tuples of jets of constant maps. Note that this condition does not depend on the chosen holomorphic coordinates near the marked points, as it is equivalent to the condition that  $u$  should have vanishing derivatives up to order  $\ell_i$  at  $\zeta_i$  for each  $i = 1, \dots, m$ . Whenever the regularity condition is satisfied,  $\text{ev}^{-1}(Z) \subset \bar{\partial}_J^{-1}(0)$  inherits the structure of a

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<sup>5</sup>The smoothness of  $\text{ev}$  is clear because it is the restriction to  $\bar{\partial}_J^{-1}(0)$  of a map  $\mathcal{B} \rightarrow \text{Jet}_J^{\ell_1}(M) \times \dots \times \text{Jet}_J^{\ell_m}(M)$  which in the natural Banach manifold charts provided by [Eli67] looks like a linear map evaluating derivatives of functions at the fixed points  $\Theta \subset \Sigma$ . This works because we are choosing to represent elements of  $\mathcal{M}_{g,m}(A, J)$  by maps with marked points at fixed positions; of course there is no actual constraint on the movement of the marked points, but this freedom is seen in our setup by varying  $j$  in  $\mathcal{T}$  instead of varying the points  $\zeta_1, \dots, \zeta_m$ . This is a notable difference from the setup in [OZ09].

smooth submanifold with real codimension  $2n \sum_i \ell_i$ , so  $\widehat{\mathcal{M}}_{g,m}(A, J; \ell_1, \dots, \ell_m)$  in general becomes an orbifold near  $[(\Sigma, j_0, \Theta, u_0)]$ , with

$$\begin{aligned} \dim \widehat{\mathcal{M}}_{g,m}(A, J; \ell_1, \dots, \ell_m) &= \dim \mathcal{M}_{g,m}(A, J) - 2n \sum_i \ell_i = \dim \mathcal{M}_g(A, J) + 2m - 2n \sum_i \ell_i \\ &= \dim \mathcal{M}_g(A, J) - \sum_{i=1}^m (2n\ell_i - 2). \end{aligned}$$

To prove that the constrained regularity condition can be achieved generically, fix  $J_{\text{ref}} \in \mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$  and a suitable sequence of positive numbers  $\varepsilon_\nu \rightarrow 0$ , and consider a Banach manifold  $\mathcal{J}_\varepsilon$  of almost complex structures in  $\mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$  that are  $C_\varepsilon$ -close to  $J_{\text{ref}}$  (cf. §6). This gives rise to two universal moduli spaces,

$$\mathcal{U}^*(\mathcal{J}_\varepsilon) := \{(u, J) \mid J \in \mathcal{J}_\varepsilon \text{ and } u \in \mathcal{M}_{g,m}^*(A, J)\}$$

and

$$\widehat{\mathcal{U}}^*(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m) := \{(u, J) \mid J \in \mathcal{J}_\varepsilon \text{ and } u \in \widehat{\mathcal{M}}_{g,m}^*(A, J; \ell_1, \dots, \ell_m)\},$$

where we abbreviate by

$$\mathcal{M}_{g,m}^*(A, J) \subset \mathcal{M}_{g,m}(A, J) \quad \text{and} \quad \widehat{\mathcal{M}}_{g,m}^*(A, J; \ell_1, \dots, \ell_m) \subset \widehat{\mathcal{M}}_{g,m}(A, J; \ell_1, \dots, \ell_m)$$

the subspaces defined via the condition that  $u$  be somewhere injective and pass through  $\mathcal{U}$ . As is well known,  $\mathcal{U}^*(\mathcal{J}_\varepsilon)$  is a separable and metrizable smooth Banach manifold, and for  $[(\Sigma, j_0, \Theta, u_0)] \in \mathcal{M}_{g,m}^*(A, J_0)$ , a neighborhood of  $(u_0, J_0)$  in  $\mathcal{U}^*(\mathcal{J}_\varepsilon)$  can be identified with the zero-set of a smooth section

$$\bar{\partial} : \mathcal{T} \times \mathcal{B} \times \mathcal{J}_\varepsilon \rightarrow \mathcal{E} : (j, u, J) \mapsto \bar{\partial}_J(u),$$

where  $\mathcal{E}$  now denotes the Banach space bundle with fibers

$$\mathcal{E}_{(j,u,J)} = W^{k-1,p}(\overline{\text{Hom}}_{\mathbb{C}}((T\Sigma, j), (u^*TM, J))).$$

The tangent space  $T_{(u_0, J_0)} \mathcal{U}^*(\mathcal{J}_\varepsilon)$  is the kernel of the surjective operator

$$\begin{aligned} \mathbf{L} := D\bar{\partial}(j_0, u_0, J_0) : T_{j_0} \mathcal{T} \oplus W^{k,p}(u_0^*TM) \oplus T_{J_0} \mathcal{J}_\varepsilon &\rightarrow W^{k-1,p}(\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, u_0^*TM)) \\ (y, \eta, Y) &\mapsto J_0 \circ Tu_0 \circ y + \mathbf{D}_{u_0} \eta + Y \circ Tu_0 \circ j_0, \end{aligned}$$

where  $\mathbf{D}_{u_0}$  is the linearized Cauchy-Riemann operator associated to  $u_0 : (\Sigma, j_0) \rightarrow (M, J_0)$ . We can again define the smooth jet evaluation map

$$(A.2) \quad \text{ev} : \bar{\partial}^{-1}(0) \rightarrow \text{Jet}_J^{\ell_1}(M) \times \dots \times \text{Jet}_J^{\ell_m}(M)$$

and identify a neighborhood of  $(u_0, J_0)$  in  $\widehat{\mathcal{U}}^*(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m)$  with  $\text{ev}^{-1}(Z)$ . The main technical ingredient behind Proposition A.1 is now the following.

**Lemma A.3.** *The jet evaluation map (A.2) is a submersion.*

*Proof.* We need to show that for any  $X \in T_{\text{ev}(u_0)}(\text{Jet}_J^{\ell_1}(M) \times \dots \times \text{Jet}_J^{\ell_m}(M))$ , there exists an element  $(y, \eta, Y) \in \ker \mathbf{L}$  with

$$d\text{ev}(u_0)\eta = X.$$

Let us first observe that this problem can be solved locally near the marked points: in fact, there exists a smooth section  $\eta \in \Gamma(u_0^*TM)$  with

$$\mathbf{D}_{u_0}\eta = 0 \text{ near } \Theta \quad \text{and} \quad \text{dev}(u_0)\eta = X.$$

This follows from the local existence theorem for  $J$ -holomorphic curves with prescribed holomorphic derivatives at a point, cf. [Wen, Theorem 2.100]. More precisely, choose a smooth path  $\gamma = (\gamma_1, \dots, \gamma_m) : (-\delta, \delta) \rightarrow \text{Jet}_J^{\ell_1}(M) \times \dots \times \text{Jet}_J^{\ell_m}(M)$  with  $\gamma(0) = \text{ev}(u_0)$  and  $\dot{\gamma}(0) = X$ . Then the local existence theorem provides for each  $i = 1, \dots, m$  a smooth family of  $J$ -holomorphic curves  $u_\tau^{(i)} : \mathbb{D}_\epsilon \rightarrow M$  defined on sufficiently small disks  $\mathbb{D}_\epsilon \subset \mathbb{C}$  such that the holomorphic  $\ell_i$ -jet represented by  $u_\tau^{(i)}$  is  $\gamma_i(\tau)$  for each  $\tau$ . The desired section  $\eta \in \Gamma(u_0^*TM)$  can now be constructed by writing it in our chosen holomorphic coordinates near each marked point  $\zeta_i$  as  $\partial_\tau u_\tau^{(i)}|_{\tau=0}$  and then extending it arbitrarily outside these neighborhoods.

Given  $\eta$  as above, we aim now to find a pair  $(\xi, Y) \in W^{k,p}(u_0^*TM) \oplus T_{J_0}\mathcal{J}_\epsilon$  such that

$$\mathbf{L}(0, \eta + \xi, Y) = \mathbf{L}(0, \xi, Y) + \mathbf{D}_{u_0}\eta = 0 \quad \text{and} \quad \text{dev}(u_0)\xi = 0,$$

in which case  $(0, \eta + \xi, Y) \in T_{(u_0, J_0)}\mathcal{W}^*(\mathcal{J}_\epsilon)$  and  $\text{dev}(u_0, J_0)(0, \eta + \xi, Y) = X$ . We will use the weighted Sobolev spaces described in §3.2. Let  $\dot{\Sigma} := \Sigma \setminus \Theta$ , and assume without loss of generality that  $u_0^{-1}(\mathcal{U}) \subset \Sigma$  is disjoint from  $\Theta$ ; this can be achieved at the cost of shrinking  $\mathcal{U}$  and therefore the space of perturbations  $\mathcal{J}_\epsilon$ . As a consequence,  $Y \circ Tu_0 \circ j_0$  now has compact support in  $\dot{\Sigma}$  for any  $Y \in T_{J_0}\mathcal{J}_\epsilon$ . Using the fixed holomorphic coordinates on neighborhoods of marked points  $\zeta_i \in \Theta$ , we can identify them biholomorphically with half-cylinders  $[0, \infty) \times S^1$  and fix trivializations of  $u_0^*TW$  on these neighborhoods to define weighted Sobolev norms and a bounded linear map

$$\dot{\mathbf{D}}_{u_0} : W^{k,p,\delta}(u_0^*TM|_{\dot{\Sigma}}) \rightarrow W^{k-1,p,\delta}(\overline{\text{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u_0^*TM)|_{\dot{\Sigma}}),$$

where sections  $\eta$  of class  $W^{k,p,\delta}$  are required to satisfy  $e^{\delta s}\eta \in W^{k,p}([0, \infty) \times S^1)$  when expressed in the chosen trivialization and holomorphic coordinates  $(s, t) \in [0, \infty) \times S^1$  on each cylindrical end near  $\Theta$ . As explained in §3.2,  $\dot{\mathbf{D}}_{u_0}$  is asymptotic to the trivial asymptotic operator at each puncture and is thus Fredholm for any  $\delta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ . We claim that whenever this condition is satisfied, the linear map

$$\begin{aligned} \mathbf{L}_\delta : W^{k,p,\delta}(u_0^*TM|_{\dot{\Sigma}}) \oplus T_{J_0}\mathcal{J}_\epsilon &\rightarrow W^{k-1,p,\delta}(\overline{\text{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u_0^*TM)|_{\dot{\Sigma}}) \\ (\xi, Y) &\mapsto \dot{\mathbf{D}}_{u_0}\xi + Y \circ Tu_0 \circ j_0 \end{aligned}$$

is surjective. The proof is more or less standard: we start with the case  $k = 1$  and note that since  $\dot{\mathbf{D}}_{u_0}$  is Fredholm,  $\mathbf{L}_\delta$  has closed range, so it is not surjective if and only if there exists a nontrivial section  $\lambda \in (L^{p,\delta})^* = L^{q,-\delta}$  for  $1/p + 1/q = 1$  which is  $L^2$ -orthogonal to the images of both  $\eta \mapsto \dot{\mathbf{D}}_{u_0}\eta$  and  $Y \mapsto Y \circ Tu_0 \circ j_0$ . Since  $u_0$  has an injective point  $z_0 \in \dot{\Sigma}$  with  $u(z_0) \in \mathcal{U}$ , the latter implies that  $\lambda$  vanishes near  $z_0$ , while the former implies that it is a weak solution to the formal adjoint equation  $\dot{\mathbf{D}}_{u_0}^* \lambda = 0$  and is therefore smooth with isolated zeroes, giving a contradiction. The case of general  $k \in \mathbb{N}$  follows from this via elliptic regularity, namely Lemma 3.12.

With this claim in place, we observe that  $-\mathbf{D}_{u_0}\eta$  vanishes near  $\Theta$  and thus restricts to  $\dot{\Sigma}$  as a section of class  $W^{k-1,p,\delta}$  for any  $\delta > 0$ , thus we can find  $\xi \in W^{k,p,\delta}(u_0^*TM|_{\dot{\Sigma}})$  and

$Y \in T_{J_0} \mathcal{J}_\varepsilon$  such that

$$\mathbf{L}(0, \xi, Y) = -\mathbf{D}_{u_0} \eta \quad \text{on} \quad \dot{\Sigma}.$$

Since  $Y$  has compact support in  $\dot{\Sigma}$  and  $\mathbf{D}_{u_0} \eta = 0$  near  $\Theta$ , this equation implies  $\mathbf{D}_{u_0} \xi = 0$  near  $\Theta$ . The continuous inclusion  $W^{k,p,\delta} \hookrightarrow C^0$  implies that  $\xi$  also has a continuous extension over  $\Sigma$  that vanishes on  $\Theta$ ; moreover, since (A.1) implies a continuous inclusion  $W^{k,p} \hookrightarrow C^1$ ,  $\xi$  has a bounded first derivative on the cylindrical ends, implying via a short computation that for  $1 < q < 2$ , the  $L^q$ -norm of its derivative on punctured disk-like neighborhoods of  $\Theta$  is finite. It follows that the extension of  $\xi$  over the punctures is in  $W^{1,q}$  on  $\Sigma$ , and elliptic regularity then implies that it is smooth everywhere. Finally, the exponential weight condition implies that in each holomorphic coordinate system identifying the neighborhood of a marked point  $\zeta_i \in \Theta$  with  $\mathbb{D}$  such that  $\zeta_i$  is at the origin, we have

$$|\xi(z)| \leq c|z|^{\delta/2\pi}$$

for some constant  $c > 0$ . But the choice of  $\delta > 0$  in this discussion was arbitrary, so choosing it large enough, we can arrange for  $\xi$  to have vanishing derivatives of arbitrarily large finite order at  $\Theta$ , proving  $d\text{ev}(u_0)\xi = 0$ .  $\square$

The lemma implies that  $\widehat{\mathcal{W}}^*(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m)$  is a separable and metrizable smooth Banach manifold, so we can now apply the Sard-Smale theorem to the projection

$$\widehat{\mathcal{W}}^*(\mathcal{J}_\varepsilon; \ell_1, \dots, \ell_m) \rightarrow \mathcal{J}_\varepsilon : (u, J) \mapsto J,$$

giving a Baire subset of  $\mathcal{J}_\varepsilon$  for which  $\widehat{\mathcal{M}}_{g,m}^*(A, J; \ell_1, \dots, \ell_m)$  is a manifold of the correct dimension, and the countable intersection of these subsets for all  $g, m, A$  and  $(\ell_1, \dots, \ell_m)$  is again comeager in  $\mathcal{J}_\varepsilon$ , proving that there is a  $C^\infty$ -dense subset of  $\mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$  for which the statement of the theorem holds. To turn this into a Baire subset of  $\mathcal{J}(M, \omega; \mathcal{U}, J_{\text{fix}})$ , one can use the standard Taubes trick (see e.g. [Wen, §4.4.2]): present  $\widehat{\mathcal{M}}_{g,m}^*(A, J; \ell_1, \dots, \ell_m)$  as a countable union of compact subsets, and associate to each one a set of regular almost complex structures, which is open by construction and dense due to the argument above, so its intersection is comeager.

*Remark A.4.* Lemma A.3 implies that for generic  $J$ , the jet evaluation map can be made transverse to any given submanifold, hence this method can be used to understand any moduli space of holomorphic curves with marked points satisfying conditions on their derivatives, e.g. the incidence conditions studied by Cieliebak and Mohnke [CM07, CM].

## APPENDIX B. SUPER-RIGID CURVES ARE ISOLATED

A statement nearly but not completely identical to Proposition 1.3 has been proved before by Zinger, see [Zin11, Prop. 3.2]. For the sake of completeness, we present a self-contained proof in this appendix which is essentially the same as Zinger's, but will be easier to follow for readers unaccustomed to the notation in his paper. Note that our statement belongs to the almost complex category and makes no reference to any symplectic structure; the proof however will introduce an auxiliary symplectic structure and use Gromov's compactness theorem.

We recall from §2.1 that if  $u \in \mathcal{M}_g(A, J)$  and  $d \geq 1$  and  $h \geq 0$  are integers, we denote by

$$\overline{\mathcal{M}}_h(d; u) \subset \overline{\mathcal{M}}_h(dA, J)$$

the moduli space of all stable nodal  $d$ -fold covers of  $u$  with arithmetic genus  $h$ .

Suppose  $J_k \rightarrow J_\infty$  is a  $C^\infty$ -convergent sequence of almost complex structures on a manifold  $M$ , and  $[(\Sigma, j_\infty, u_\infty)] \in \mathcal{M}_g(A, J_\infty)$  is a super-rigid curve. Then  $u_\infty$  is Fredholm regular with index 0, so the implicit function theorem implies the existence of curves  $u_k : (\Sigma, j_k) \rightarrow (M, J_k)$  for sufficiently large  $k$  such that  $j_k \rightarrow j_\infty$  and  $u_k \rightarrow u_\infty$  in  $C^\infty$ ; these curves are unique up to biholomorphic reparametrization, and are also simple and immersed for sufficiently large  $k$ . Assume  $v_k \in \overline{\mathcal{M}}_h(dA, J_k)$  is a sequence of  $J_k$ -holomorphic curves converging to a nodal cover  $\tilde{u} \in \overline{\mathcal{M}}_h(d; u_\infty)$  for some  $d > 0$ . We will show that if the curves  $v_k$  are not covers of  $u_k$  for all sufficiently large  $k$ , then rescaling the normal fibers near  $u_k$  as  $k \rightarrow \infty$  gives rise to a nontrivial section in the kernel of the normal Cauchy-Riemann operator on some cover of  $u_\infty$ , contradicting super-rigidity.

Choose a convergent sequence of  $J_k$ -invariant Riemannian metrics and corresponding Levi-Civita connections  $\nabla^k$ . Since the maps  $u_k$  are immersed, we can define  $J_k$ -invariant normal bundles  $N_{u_k} \rightarrow \Sigma$  as the orthogonal complements of  $\text{im } du_k$ . These are all isomorphic as real vector bundles, so we can identify them all with the real bundle  $N := N_{u_\infty} \subset u_\infty^* TM$  carrying a sequence of complex structures

$$(N, J_k) \xrightarrow{\pi} (\Sigma, j_k),$$

and then use the sequence of exponential maps determined by  $\nabla^k$  to define a  $C^\infty$ -convergent sequence of immersions

$$\Psi_k : \mathcal{N}(\Sigma) \rightarrow M$$

of some fixed neighborhood  $\mathcal{N}(\Sigma) \subset N$  of the zero section  $\Sigma \subset N$  onto some neighborhood of  $u_k(\Sigma)$ , such that  $\Psi_k|_\Sigma = u_k$ . Let  $\hat{J}_k = \Psi_k^* J_k$  for  $k = 1, 2, 3, \dots, \infty$ , so that for  $k$  sufficiently large, the curves  $v_k$  can be identified with  $\hat{J}_k$ -holomorphic curves in the total space of  $N$ , and each  $u_k$  is identified with the zero section.

Let  $\pi_N : u_\infty^* TM \rightarrow N$  denote the normal projection, so that  $\hat{\nabla} := \pi_N \circ \nabla^\infty$  induces a connection on  $N \rightarrow \Sigma$  (as a *real* vector bundle), and thus defines a splitting into horizontal and vertical subbundles

$$TN = HN \oplus VN.$$

This splitting is invariant under the diffeomorphisms on  $N$  defined by real scalar multiplication. For  $z \in \Sigma$  and  $\eta \in N_z$ , the fibers in the splitting admit canonical identifications

$$H_{(z, \eta)} N = T_z \Sigma, \quad V_{(z, \eta)} N = N_z,$$

and we can write  $\hat{J}_k$  with respect to the splitting as

$$\hat{J}_k(z, \eta) = \begin{pmatrix} \alpha_k(z, \eta) & \beta_k(z, \eta) \\ \gamma_k(z, \eta) & \delta_k(z, \eta) \end{pmatrix},$$

for some smoothly varying linear maps  $\alpha_k(z, \eta) : T_z \Sigma \rightarrow T_z \Sigma$ ,  $\beta_k(z, \eta) : N_z \rightarrow T_z \Sigma$  and so forth. Since  $u_k : (\Sigma, j_k) \rightarrow (M, J_k)$  is  $J_k$ -holomorphic and the fibers of  $N_{u_k}$  are  $J_k$ -invariant along  $u_k$ , we have

$$\alpha_k(z, 0) = j_k(z), \quad \delta_k(z, 0) = J_k(u_k(z)), \quad \beta_k(z, 0) = 0, \quad \gamma_k(z, 0) = 0.$$



Now for any constant  $r > 0$ , the diffeomorphism

$$\Phi_r : N \rightarrow N : (z, \eta) \mapsto (z, r\eta)$$

transforms  $\hat{J}_k$  to

$$\hat{J}_k^r(z, \eta) := \Phi_r^* \hat{J}_k|_{(z, \eta)} = \begin{pmatrix} \alpha_k(z, r\eta) & r\beta_k(z, r\eta) \\ \frac{1}{r}\gamma_k(z, r\eta) & \delta_k(z, r\eta) \end{pmatrix},$$

so given any positive sequence  $r_k \rightarrow 0$ , the sequence  $\hat{J}_k^{r_k}$  converges in  $C^\infty$  on compact subsets of  $N$  to

$$(B.1) \quad \hat{J}_\infty^0(z, \eta) := \begin{pmatrix} j_\infty(z) & 0 \\ d\gamma_\infty(z, 0)\eta & J_\infty(u_\infty(z)) \end{pmatrix}.$$

**Lemma B.1.** *A neighborhood of  $\Sigma$  in  $N$  admits a symplectic form  $\omega$  that tames  $\hat{J}_\infty^0$ .*

*Proof.* We use a variation on Thurston's method for constructing symplectic forms on fibrations (cf. [MS98, Theorem 6.3]). For any open subset  $\mathcal{U} \subset \Sigma$ , let  $\Lambda(\mathcal{U})$  denote the space of smooth 1-forms  $\lambda$  on  $\pi^{-1}(\mathcal{U})$  satisfying the following conditions:

(i) At any point  $(z, 0) \in \mathcal{U} \subset N|_{\mathcal{U}}$  in the zero section,

$$\lambda|_{(z, 0)} = 0 \quad \text{and} \quad d\lambda|_{T_z \Sigma \times N_z} = 0;$$

(ii) The restriction of  $d\lambda$  to fibers in  $\pi^{-1}(\mathcal{U})$  defines a symplectic vector bundle structure on  $N|_{\mathcal{U}}$  taming  $J_\infty$ .

We observe that  $\Lambda(\mathcal{U})$  is nonempty whenever there exists a complex trivialization of  $(N, J_\infty)$  over  $\mathcal{U}$ , and moreover, it is  $C^\infty$ -convex in the sense that if  $\lambda_0, \lambda_1 \in \Lambda(\mathcal{U})$ , then

$$(\psi \circ \pi)\lambda_1 + (1 - \psi \circ \pi)\lambda_0 \in \Lambda(\mathcal{U})$$

for every smooth function  $\psi : \mathcal{U} \rightarrow [0, 1]$ . It follows that an element of  $\Lambda(\Sigma)$  can be constructed by patching together local constructions via a partition of unity.

Now given  $\lambda \in \Lambda(\Sigma)$ , choose an area form  $\sigma$  on  $\Sigma$  taming  $j_\infty$ . Then for a sufficiently large constant  $K > 0$ ,

$$\omega := K\pi^*\sigma + d\lambda$$

is a closed 2-form that tames  $\hat{J}_\infty^0$  at  $\Sigma$  and hence also in a neighborhood of  $\Sigma$ .  $\square$

*Remark B.2.* The above proof did not use any special properties of  $\hat{J}_\infty^0$  except that the zero section is pseudoholomorphic and the normal fibers along the zero section are also complex. The same argument shows that for any embedded closed  $J$ -holomorphic curve in any almost complex manifold  $(M, J)$ , a neighborhood of the curve admits a symplectic form that tames  $J$ .

**Lemma B.3.** *Suppose  $\psi : \tilde{\Sigma} \rightarrow \Sigma$  is a smooth map,  $\tilde{j}$  is a complex structure on  $\tilde{\Sigma}$ , and  $\xi \in \Gamma(\psi^*N)$  is a smooth section along  $\psi$ . Then the resulting map into the total space  $\xi : \tilde{\Sigma} \rightarrow N$  is a pseudoholomorphic map  $(\tilde{\Sigma}, \tilde{j}) \rightarrow (N, \hat{J}_\infty^0)$  if and only if  $\psi : (\tilde{\Sigma}, \tilde{j}) \rightarrow (\Sigma, j_\infty)$  is holomorphic and  $\xi \in \ker \mathbf{D}_{u_\infty \circ \psi}^N$ .*

*Proof.* Write  $v(z) := \xi(\psi(z))$ . Then using (B.1), the equation  $Tv + \hat{J}_\infty^0 \circ Tv \circ \tilde{j} = 0$  translates into the two equations

$$d\psi(z) + j_\infty(\psi(z)) \circ d\psi(z) \circ \tilde{j}(z) = 0,$$

and

$$\widehat{\nabla}\eta(z) + J_\infty(u_\infty(\psi(z))) \circ \widehat{\nabla}\eta(z) \circ \tilde{j} + [d\gamma_\infty(\psi(z), 0)\eta(z)] d\psi(z) \circ \tilde{j} = 0$$

for  $z \in \tilde{\Sigma}$ . The first equation says that  $\psi : (\tilde{\Sigma}, \tilde{j}) \rightarrow (\Sigma, j_\infty)$  is holomorphic, and under this assumption, the second matches  $\mathbf{D}_{u_\infty \circ \psi}^N \eta = 0$  after observing

$$[d\gamma_\infty(\psi, 0)\eta] \circ d\psi \circ \tilde{j} = \pi_N \circ (\nabla_\eta J_\infty) \circ T(u_\infty \circ \psi) \circ \tilde{j}.$$

□

We now prove Proposition 1.3 as follows. Arguing by contradiction, assume after taking a subsequence that the curves  $v_k : (\tilde{\Sigma}, \tilde{j}_k) \rightarrow (M, J_k)$  are not covers of  $u_k$  for any  $k$  as  $k \rightarrow \infty$ . Choose a symplectic form  $\omega$  near the zero section in  $N = N_{u_\infty}$  as given by Lemma B.1, and choose  $\delta > 0$  such that  $\omega$  tames  $\hat{J}_\infty^0$  on  $\{\eta \in N \mid |\eta| < 2\delta\}$ . Writing  $v_k(z) = \xi_k(\psi_k(z))$  for sequences  $\psi_k : \tilde{\Sigma} \rightarrow \Sigma$  and  $\xi_k \in \Gamma(\psi_k^* N)$ , we have

$$r_k := \frac{1}{\delta} \max_{z \in \tilde{\Sigma}} |\xi_k(z)| > 0$$

and  $r_k \rightarrow 0$  by assumption. Then

$$w_k := \Phi_{r_k}^{-1} \circ v_k : (\tilde{\Sigma}, \tilde{j}_k) \rightarrow (N, \hat{J}_k^{r_k})$$

is a sequence of smooth pseudoholomorphic curves in a compact subset of the neighborhood  $\{\eta \in N \mid |\eta| < 2\delta\}$ , which can be written as  $w_k(z) = \eta_k(\psi_k(z))$  where  $\eta_k = \frac{1}{r_k} \xi_k$  satisfies

$$(B.2) \quad \max_{z \in \tilde{\Sigma}} |\eta_k(z)| = \delta.$$

Note that since  $v_k$  converges to a nodal curve in  $\overline{\mathcal{M}}_h(d; u_\infty)$ , we can also assume the maps  $\psi_k : \tilde{\Sigma} \rightarrow \Sigma$  have fixed degree  $d$ . Then since  $\hat{J}_k^{r_k} \rightarrow \hat{J}_\infty^0$  and the latter is tamed by  $\omega$  in the region under consideration, Gromov compactness applies to  $w_k$  and yields a subsequence convergent to a stable nodal curve  $w_\infty \in \overline{\mathcal{M}}_h(d[\Sigma], \hat{J}_\infty^0)$ . By Lemma B.3, each smooth component  $w$  of  $w_\infty$  has the form  $w(z) = \eta(\psi(z))$  where  $\psi : (\tilde{\Sigma}, \tilde{j}) \rightarrow (\Sigma, j_\infty)$  is holomorphic and  $\mathbf{D}_{u_\infty \circ \psi}^N \eta = 0$ . We claim there must be at least one such component for which  $\deg(\psi) > 0$  and  $\eta \neq 0$ . Indeed, (B.2) implies that there is at least one component with  $\eta \neq 0$ . If every such component also satisfies  $\deg(\psi) = 0$ , then  $\eta$  is a nonzero constant on this component, as the normal operator  $\mathbf{D}_{u_\infty \circ \psi}^N$  is simply the standard Cauchy-Riemann operator on a trivial bundle when  $\psi$  is constant. But since  $\deg(\psi_k) = d > 0$ , any component with  $\deg(\psi) = 0$  is necessarily connected by a chain of nodes to another component with  $\deg(\psi) > 0$ , and on this component,  $\eta$  is nonzero at the nodal point. This implies the existence of a nontrivial element  $\eta \in \ker \mathbf{D}_{u_\infty \circ \psi}^N$  for some positive degree holomorphic cover  $\psi$ , and thus violates super-rigidity. The proof of Proposition 1.3 is complete.

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